

# Four-point renormalized coupling constant and Callan-Symanzik $\beta$ -function in $O(N)$ models.

Andrea Pelissetto and Ettore Vicari

*Dipartimento di Fisica dell'Università and I.N.F.N., I-56126 Pisa, Italy*

(February 1, 2008)

## Abstract

We investigate some issues concerning the zero-momentum four-point renormalized coupling constant  $g$  in the symmetric phase of  $O(N)$  models, and the corresponding Callan-Symanzik  $\beta$ -function.

In the framework of the  $1/N$  expansion we show that the Callan-Symanzik  $\beta$ -function is non-analytic at its zero, i.e. at the fixed-point value  $g^*$  of  $g$ .

This fact calls for a check of the actual accuracy of the determination of  $g^*$  from the resummation of the  $d = 3$  perturbative  $g$ -expansion, which is usually performed assuming the analyticity of the  $\beta$ -function. Two alternative approaches are exploited.

We extend the  $\epsilon$ -expansion of  $g^*$  to  $O(\epsilon^4)$ . Quite accurate estimates of  $g^*$  are obtained by an analysis that exploits the analytic behavior of  $g^*$  as a function of  $d$  and the known values of  $g^*$  for lower-dimensional  $O(N)$  models, i.e. for  $d = 2, 1, 0$ .

Accurate estimates of  $g^*$  are also obtained by a reanalysis of the strong-coupling expansion of the lattice  $N$ -vector model allowing for the leading confluent singularity.

The agreement among the  $g$ -,  $\epsilon$ -, and strong-coupling expansion results is good for all values of  $N$ . However, at  $N = 0, 1$ ,  $\epsilon$ - and strong-coupling expansion favor values of  $g^*$  which are slightly lower than those obtained by the resummation of the  $g$ -expansion assuming analyticity in the Callan-Symanzik  $\beta$ -function.

**Keywords:** Field theory, Critical phenomena,  $O(N)$  models, Four-point renormalized coupling constant, Perturbative expansion at fixed dimension,  $1/N$ -expansion,  $\epsilon$ -expansion, Strong-coupling expansion.

**PACS numbers:** 11.10.Kk, 11.15.Pg, 11.15.Me, 64.60.Fr, 75.10.Hk.

## I. INTRODUCTION

The renormalization-group theory of critical phenomena provides a description of statistical models in the neighbourhood of the critical point. For  $O(N)$  models calculations are based on the  $\phi^4$ -field theory defined by the action

$$S = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4!} g_0 (\phi^2)^2 \right]. \quad (1)$$

A strategy, which has been largely employed in the study of the symmetric phase, relies on a perturbative expansion in powers of the zero-momentum four-point renormalized coupling constant  $g$  performed at fixed dimension  $d = 3$  [1]. This perturbative expansion is asymptotic; nonetheless accurate results can be obtained by resummations exploiting its Borel summability and the knowledge of the large-order behavior. As general references on the  $g$ -expansion method see for instance Refs. [2–4]. This technique has led to accurate estimates of the critical exponents.

An important quantity entering the calculation of universal quantities is the fixed-point value of  $g$ , i.e. the zero of the corresponding Callan-Symanzik  $\beta$ -function. In the critical region, the bare coupling constant  $g_0$  becomes infinite on the scale fixed by the correlation length, whereas the zero-momentum four-point renormalized coupling approaches a finite non-zero limit  $g^*$  at criticality. Accurate calculations of  $g^*$  have been done by analyzing the perturbative expansion of the Callan-Symanzik  $\beta$ -function [5–10] (known to  $O(g^7)$ ), using the known results for its large-order behavior. The best determinations of  $g^*$  have been apparently obtained by Le Guillou and Zinn-Justin [6] by making some additional assumptions on the analytic properties of the Borel transform. Such additional assumptions have been questioned by Nickel [11,7], who argued the presence of confluent singularities in the  $\beta$ -function at its zero, which may complicate the analytic structure of the Borel transform.

This issue needs a non-perturbative analysis to be clarified, furthermore an analytic approach is required in order to understand the nature of the singularities. These features can be realized in the framework of the  $1/N$  expansion. In this paper we analyze the Callan-Symanzik  $\beta$ -function computed in Ref. [12] to  $O(1/N)$  (i.e. the next-to-leading order). While the leading order is analytic, the  $O(1/N)$  term shows the presence of confluent singularities at the zero of the  $\beta$ -function for all  $2 < d < 4$ . Moreover a phenomenon analogous to the Abe-Hikami anomaly for the specific-heat [13] emerges at the special dimensions  $d = 4 - 2/n$  (where  $n$  is an integer number), thus including the interesting case  $d = 3$ .

In the analysis of Ref. [6] confluent singularities at the zero of the  $\beta$ -function may cause a slow convergence to the correct fixed-point value of  $g$  [11]. The apparent stability of the results when analyzing a finite number of terms of the perturbative expansion may then not provide a reliable indication of the uncertainty of the overall estimate. Confluent singularities represent a source of systematic error for the procedure used in Ref. [6]. A more general analysis explicitly allowing for the presence of confluent singularities would slightly change the value of  $g^*$  for small values of  $N$  (although not excluding the values obtained in Ref. [6]) and consequently the values of the critical exponents [8]. It is therefore important to exploit other approaches to the study of  $O(N)$  models, which can provide a check of the estimates of  $g^*$  from the resummations of the perturbative  $g$ -expansion, and of their actual accuracy.

An alternative field-theoretic strategy is the expansion in powers of  $\epsilon = 4 - d$  [14]. An important advantage of the  $\epsilon$ -expansion is the possibility of working directly at criticality. This allows us to go from one phase to another using the same framework. In order to get estimates at  $\epsilon = 1$ , the  $\epsilon$ -expansion requires eventually a resummation which is usually performed assuming its Borel summability. Relatively long series of the critical exponents have been calculated and their analysis has led to estimates which are in substantial agreement with those determined by the  $g$ -expansion (see e.g. Ref. [15]).

The fixed-point value of the zero-momentum four-point renormalized coupling is known only to  $O(\epsilon^2)$  [16], thus not allowing a real check of the value of  $g^*$  obtained by the  $g$ -expansion. In this paper we extend this calculation to  $O(\epsilon^4)$ . The analysis of the  $O(\epsilon^4)$  series of  $g^*$  provides already good estimates with an apparent uncertainty of approximately 6% for small values of  $N$ . A considerable improvement in the analysis of the  $\epsilon$ -expansion of  $g^*$  is achieved using the known values of  $g^*$  for lower-dimensional  $O(N)$  models, i.e. for  $d = 2, 1, 0$ . The key point is that  $g^*$  is expected to be analytic and quite smooth in the domain  $0 < d < 4$  (thus  $0 < \epsilon < 4$ ). This can be indeed verified in the large- $N$  limit to  $O(1/N)$  using the results of Ref. [12]. Generalizing the technique presented in Ref. [15], we perform a polynomial interpolation among the values of  $d$  where  $g^*$  is known ( $d = 0, 1$ ) or for which good estimates are available ( $d = 2$ , especially for  $N = 0, 1$  by strong-coupling calculations), and then analyze the series of the difference. This procedure leads to much more accurate estimates of  $g^*$ , which are consistent with those obtained by the direct analysis of the original  $\epsilon$ -series, but with an apparent error of approximately one per cent ( $\sim 0.5\%$  for  $N = 1$ ). The agreement with the  $g$ -expansion estimates is good for all values of  $N$ . However it is worth anticipating that the results for  $N = 0, 1$  turn out slightly lower than the estimates given by Le Guillou and Zinn-Justin [6], thus favouring the results of the more general analysis done by Nickel [8].

Another approach which has been widely used in the study of critical phenomena is based on lattice formulations of the theory. Two main techniques have been exploited in this context: high- and low-temperature expansions and Monte Carlo simulations. In the symmetric phase the fixed-point value of the zero-momentum four-point renormalized coupling can be estimated by analyzing the strong-coupling expansion of the quantities entering its definition [17–20,12,21–23]. Studies based on Monte Carlo simulations can be found in Refs. [24–31]. The agreement with the field-theoretic estimates is substantially good, but small discrepancies have been observed, especially for small values of  $N$  [12,22,30,31]. We will show that such residual discrepancies disappear (or are largely reduced) when the leading effects of the confluent singularities are properly taken into account in the analysis of the strong-coupling expansion [32,33,23] and in the analysis of the Monte Carlo data.

The paper is organized as follows:

In Sec. II we introduce some definitions and notations used in the paper.

In Sec. III we analyze the Callan-Symanzik  $\beta$ -function calculated to  $O(1/N)$  in Ref. [12], and discuss the presence of confluent singularities at its zero.

In Sec. IV we analyze the expansion to  $O(\epsilon^4)$  of the fixed-point value of the zero-momentum renormalized coupling constant.

Sec. V is dedicated to a reanalysis of the strong-coupling expansion which allows for the confluent singularities present in the lattice approach.

In Sec. VI some conclusions are drawn.

In App. A we give some technical details on the analysis of the analiticity properties of the Callan-Symanzik  $\beta$ -function to  $O(1/N)$ .

In App. B we present the perturbative calculation to  $O(\epsilon^4)$  of  $g^*$ .

## II. DEFINITIONS AND NOTATIONS

Let us introduce a few definitions and notations. If we set

$$\langle \phi^\alpha(0)\phi^\beta(x) \rangle = \delta^{\alpha\beta}G^{(2)}(x), \quad (2)$$

$$\langle \phi^\alpha(0)\phi^\beta(x)\phi^\gamma(y)\phi^\delta(z) \rangle_c = \frac{1}{3}(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma})G^{(4)}(x,y,z), \quad (3)$$

the zero-momentum four-point renormalized coupling  $g$  is defined as

$$g \equiv -\frac{\int dx dy dz G^{(4)}(x,y,z)}{\xi^d [\int dx G^{(2)}(x)]^2}, \quad (4)$$

where  $\xi$  is the second-moment correlation length

$$\xi^2 = \frac{1}{2d} \frac{\int dx x^2 G^{(2)}(x)}{\int dx G^{(2)}(x)}. \quad (5)$$

The normalization in Eq. (4) is such that in perturbation theory  $g = g_0/m_0^\epsilon + O(g_0^2/m_0^{2\epsilon})$ .

For convenience in the paper we will also introduce other definitions. First of all we consider the rescaled coupling

$$\bar{g} \equiv \frac{1}{2(4\pi)^{d/2}} \frac{(N+8)}{3} \Gamma\left(2 - \frac{d}{2}\right) g. \quad (6)$$

Unlike  $g^*$  which is of order  $(4-d) = \epsilon$  for  $d \rightarrow 4$ , the fixed-point value of  $\bar{g}$  is  $O(1)$ , due to the factor multiplying  $g$  in its definition. Moreover  $\bar{g}$  has the property that for  $N \rightarrow \infty$ ,  $\bar{g}^* \rightarrow 1$  for any dimension  $d$ . For  $d = 3$ , it coincides with the coupling which is usually used in the analysis of the perturbative expansions in fixed dimension  $d = 3$  [5,6]. In the large- $N$  limit another definition is also useful:

$$\hat{g} \equiv \frac{Ng}{3}. \quad (7)$$

Finally a fourth definition is quite common in the literature:

$$f \equiv \frac{N+2}{3}g. \quad (8)$$

The quantity  $f$  is naturally defined in terms of

$$\chi \equiv \int dx \langle \phi(0) \cdot \phi(x) \rangle, \quad (9)$$

$$\chi_4 \equiv \int dx dy dz \langle \phi(0) \cdot \phi(x) \phi(y) \cdot \phi(z) \rangle_c, \quad (10)$$

as

$$f \equiv -N \frac{\chi_4}{\chi^2 \xi^d}. \quad (11)$$

It is well-known that the  $\lambda\phi^4$  model defined in Eq. (1) is equivalent at criticality to the  $N$ -vector model defined in the continuum by

$$S = \frac{\beta N}{2} \int d^d x \partial_\mu s(x) \cdot \partial_\mu s(x) \quad (12)$$

where  $s(x) \cdot s(x) = 1$ .

On the lattice one can consider any discretization with the (formal) continuum limit given by Eq. (12). Here we will study the theory with nearest-neighbour interactions defined by

$$S_L = -\beta N \sum_{\langle xy \rangle} s(x) \cdot s(y), \quad (13)$$

where the sum extends over all lattice links  $\langle xy \rangle$ . In this case the renormalized coupling constant is given by the previous formulae with the obvious substitution  $\phi \rightarrow s$ . Of course, on the lattice, the integrals are replaced by sums over the lattice points.

### III. CRITICAL-POINT NON-ANALYTICITY IN THE LARGE- $N$ LIMIT

An important controversial issue in the field-theory method at fixed dimension  $d = 3$  is the presence of non-analyticities at the critical point  $g^*$ . The question was raised long ago by Nickel [11] who gave a simple argument to show that non-analytic terms should in principle be present in the  $\beta$ -function. The same argument applies also to other series, like those defining the critical exponents: any quantity should be expected to be singular at the critical point.

To understand the problem, let us consider the four-point renormalized coupling  $g$  as a function of the temperature  $T$ . For  $T \rightarrow T_c$  we can write down an expansion of the form

$$g = g^* \left[ 1 + a_1(T - T_c) + a_2(T - T_c)^2 + \dots + b_1(T - T_c)^\Delta + b_2(T - T_c)^{2\Delta} + \dots + c_1(T - T_c)^{\Delta+1} + \dots + d_1(T - T_c)^{\Delta_2} + \dots + e_1(T - T_c)^{\Delta_3} + \dots \right] \quad (14)$$

where  $\Delta, \Delta_2, \dots$  are subleading exponents. We expect on general grounds that  $a_1 = a_2 = a_3 = \dots = 0$ . Indeed these analytic corrections arise from the non-linearity of the scaling fields and their effect can be eliminated in the Green's functions by an appropriate change of variables [34]. For dimensionless renormalization-group invariant quantities such as  $g$ , the leading term is universal and therefore independent of the scaling fields, so that no analytic term can be generated. We will explicitly show their absence in the following computation. Notice that analytic correction factors to the singular correction terms are generally present, and therefore the constants  $c_i$  in Eq. (14) are expected to be nonzero. Moreover we mention

that, in general, the correction terms can include powers of the critical exponents<sup>1</sup>  $1 - \alpha$  and  $\gamma$  [34] (associated with backgrounds).

Starting from Eq. (14) it is easy to compute the  $\beta$ -function:

$$\beta(g) = M \frac{dg}{dM} = \frac{M}{dM/dT} \frac{dg}{dT}. \quad (15)$$

Since the mass gap  $M$  scales as

$$M \sim (T - T_c)^\nu \left[ 1 + \bar{a}_1(T - T_c) + \dots + \bar{b}_1(T - T_c)^\Delta + \dots \right], \quad (16)$$

we obtain the following expansion:

$$\begin{aligned} \beta(g) = & \alpha_1(g^* - g) + \alpha_2(g^* - g)^2 + \dots + \beta_1(g^* - g)^{\frac{1}{\Delta}} + \beta_2(g^* - g)^{\frac{2}{\Delta}} + \dots + \\ & \gamma_1(g^* - g)^{1+\frac{1}{\Delta}} + \dots + \delta_1(g^* - g)^{\frac{\Delta_2}{\Delta}} + \dots + \zeta_1(g^* - g)^{\frac{\Delta_3}{\Delta}} + \dots \end{aligned} \quad (17)$$

It is easy to verify the well-known fact that  $\alpha_1 = -\Delta/\nu \equiv -\omega$  and that, if  $a_1 = a_2 = \dots = 0$  in Eq. (14), then  $\beta_1 = \beta_2 = \dots = 0$ . Eqs. (14) and (17) are the expressions which are expected on the basis of the standard renormalization-group picture. They express the asymptotic behaviour for generic models and indeed this is the supposed behaviour in lattice theories. However for the continuum  $\lambda\phi^4$  a much simpler expansion is often conjectured<sup>2</sup> [38,39]. First of all one assumes that the continuum theory couples only to one subleading scaling field, the operator associated to the exponent  $\Delta$ . As a consequence in the expansion (14) no corrections with exponents  $\Delta_2, \Delta_3, \dots$ , should appear. In Eq. (17) this conjecture implies for instance that  $\delta_1 = \zeta_1 = 0$ , i.e. no terms with exponents  $\Delta_i/\Delta$  are present. The second claim is that  $T - T_c \equiv m_0^2 - m_{0c}^2$  is a scaling field in the Wilson renormalization-group sense [40]. Therefore in Eq. (14) no term with exponent  $h\Delta + k$ ,  $k > 0$  should appear. Correspondingly in Eq. (17) terms of the form  $h + k/\Delta$  would be absent. As a consequence  $\beta(g)$  would be analytic. Of course, these two hypotheses would also prove that also other expansions in  $g$ , like the series for the critical exponents, would be analytic at the critical point.

In order to understand the validity of these conjectures, one must compute the  $\beta$ -function non-perturbatively. The only case in which this is possible is the large- $N$  limit, i.e. in the framework of the  $1/N$  expansion.

In the following we will analyze the  $\beta$ -function computed in Ref. [12] for  $N \rightarrow \infty$  and we will show explicitly that the conjecture mentioned above is incorrect: non-analytic terms are indeed present. However, at the order of  $1/N$  we are working, we will be unable to

<sup>1</sup> In the following large- $N$  analysis we will not consider these terms. Since for large values of  $N$ ,  $\gamma/\nu = (1 - \alpha)/\nu = 2$ , their contributions mix with the analytic background, so that their inclusion would not change our main conclusions.

<sup>2</sup>For a critique of these conjectures from the point of view of the renormalization group à la Wilson, see Ref. [35], Sec. 5.2 and App. E of Ref. [36]. See also Ref. [37] for a more recent discussion in the same framework.

distinguish which hypothesis is false. Indeed since, at  $N = \infty$ ,  $1 + 1/\Delta = \Delta_2/\Delta = \Delta_3/\Delta$ , we will only be able to verify that at least one of the corresponding terms is present but not to prove the presence of all of them. This problem can only be solved by computing the next two orders in  $1/N$ .

The starting point is the  $\beta$ -function expanded in powers of  $1/N$ :

$$\beta(\hat{g}) = M \frac{d\hat{g}}{dM} = \beta^{(0)}(\hat{g}) + \frac{1}{N} \beta^{(1)}(\hat{g}) + O\left(\frac{1}{N^2}\right), \quad (18)$$

where, for convenience, we have introduced the coupling  $\hat{g}$  defined by Eq. (7). The two functions  $\beta^{(0)}(\hat{g})$  and  $\beta^{(1)}(\hat{g})$  were computed in Ref. [12]. One has

$$\beta^{(0)}(\hat{g}) = (d - 4)\hat{g} \left(1 - \frac{\hat{g}}{\hat{g}_\infty^*}\right), \quad (19)$$

and

$$\begin{aligned} \frac{\beta^{(1)}(\hat{g})}{\hat{g}^2} &= (d - 3)2^{d-1}\beta_0 + \frac{2}{d}(d - 1)^2(d - 4 + \beta_0\hat{g})^2 \int \frac{d^d u}{(2\pi)^d} \frac{1}{[1 + \hat{g}\Pi(u)]^2} \frac{1}{(4 + u^2)^2} \\ &+ 2 \int \frac{d^d u}{(2\pi)^d} \left[ \frac{\beta_0\hat{g} + d - 4}{(1 + \hat{g}\Pi(u))^2} - \frac{\beta_0\hat{g}}{1 + \hat{g}\Pi(u)} \right] \left[ \frac{1}{(1 + u^2)^3} + \frac{3}{(1 + u^2)(4 + u^2)} \left( \frac{\frac{d}{4} - 2}{1 + u^2} - \frac{d - 1}{4 + u^2} \right) \right] \\ &- \frac{2}{d} \int \frac{d^d u}{(2\pi)^d} \frac{(\beta_0\hat{g} + d - 4)^2}{(1 + \hat{g}\Pi(u))^2} \left[ \frac{1}{(1 + u^2)^3} + \frac{3}{(1 + u^2)(4 + u^2)} \left( \frac{\frac{d}{4} - 1}{1 + u^2} + \frac{d - 1}{4 + u^2} \right) \right] \\ &- 4 \int \frac{d^d u}{(2\pi)^d} \left[ \frac{\beta_0\hat{g} + d - 4}{(1 + \hat{g}\Pi(u))^3} - \frac{\beta_0\hat{g} + \frac{d}{2} - 2}{(1 + \hat{g}\Pi(u))^2} \right] \frac{1}{(4 + u^2)^2} \left( \beta_0\hat{g} + d - 4 - \frac{3}{1 + u^2} \right)^2 \\ &- 4 \int \frac{d^d u}{(2\pi)^d} \frac{\beta_0\hat{g}(\beta_0\hat{g} + d - 4)}{(1 + \hat{g}\Pi(u))^2} \frac{1}{(4 + u^2)^2} \left( \beta_0\hat{g} + d - 4 - \frac{3}{1 + u^2} \right). \end{aligned} \quad (20)$$

Here  $\hat{g}_\infty^*$  is the critical value of  $\hat{g}$  for  $N = \infty$

$$\hat{g}_\infty^* = \frac{2(4\pi)^{d/2}}{\Gamma(2 - d/2)}, \quad (21)$$

$\beta_0$  is defined by

$$\beta_0 \equiv \frac{4 - d}{\hat{g}_\infty^*}, \quad (22)$$

and

$$\Pi(u) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + 1} \left[ \frac{1}{(p + u)^2 + 1} - \frac{1}{p^2 + 1} \right]. \quad (23)$$

The leading term, Eq. (19), is clearly analytic. In App. A we study the behaviour of  $\beta^{(1)}(\hat{g})$  for  $\hat{g} \rightarrow \hat{g}_\infty^*$ . Setting

$$\Theta \equiv \frac{\hat{g}_\infty^* - \hat{g}}{\hat{g}}, \quad (24)$$

for  $2 < d < 4$ , we find

$$\frac{\beta^{(1)}(\hat{g})}{\hat{g}^2} = \text{analytic terms} + A(d)\Theta^{\frac{2}{4-d}} + \dots, \quad (25)$$

where

$$A(d) = -\left(\frac{1}{c}\right)^{1+\frac{2}{4-d}} \frac{2^{d-4}(9d^4 - 112d^3 + 428d^2 - 512d + 192)}{8d(6-d)(4-d)} \frac{\pi N_d}{\sin(2\pi/(4-d))}, \quad (26)$$

and

$$c = \frac{\sqrt{\pi}}{2} \frac{\Gamma(d/2 - 1)}{\Gamma((d-1)/2)}, \quad (27)$$

$$N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}. \quad (28)$$

This expansion is not valid whenever  $d = 4 - 2/n$ ,  $n \in \mathbb{N}$ , as in this case  $A(d)$  diverges. For these values of the dimension one finds

$$\frac{\beta^{(1)}(\hat{g})}{\hat{g}^2} = \text{analytic terms} + B(n)\Theta^n \log \Theta + \dots \quad (29)$$

where

$$B(n) = \frac{2}{(4-d)^2} \lim_{\epsilon \rightarrow 0} [\epsilon A(d+\epsilon)]. \quad (30)$$

In particular for  $d = 3$ , which corresponds to  $n = 2$ , we have

$$B(2) = -\frac{71}{12\pi^5}. \quad (31)$$

Let us now interpret the results. Let us consider first the case of generic  $d \neq 4 - 2/n$ . In the large- $N$  limit the smallest exponents (in the region  $2 < d \leq 4$ ) are [41]

$$\Delta = \frac{4-d}{d-2} + O(1/N), \quad (32)$$

$$\Delta_2 = \Delta_3 = \frac{2}{d-2} + O(1/N). \quad (33)$$

At  $N = \infty$   $\Delta_2$  and  $\Delta_3$  are degenerate independently of  $d$  [41]. This degeneracy should be lifted only at the next order<sup>3</sup>, i.e.  $O(1/N)$ . To leading order in  $1/N$  one has

$$\frac{\Delta_2}{\Delta} = \frac{\Delta_3}{\Delta} = 1 + \frac{1}{\Delta} = \frac{2}{4-d}. \quad (34)$$

<sup>3</sup> The corresponding scaling operators are linear combinations of  $O_1 \equiv (m_0^2 + \frac{1}{6}g_0\phi^2)(\nabla\phi)^2$  and  $O_2 \equiv (\nabla^2\phi)^2$  [41].

Eq. (25) shows that non-analytic terms are present. But because of the degeneracy (34), they cannot be distinguished. Correspondingly in Eq. (17) one has

$$\gamma_1 + \delta_1 + \zeta_1 \neq 0. \quad (35)$$

The evaluation to  $O(1/N)$  of the constant  $\gamma_1$ ,  $\delta_1$  and  $\zeta_1$  requires an  $O(1/N^3)$  computation. So we cannot check if some of them are zero. Notice that, as expected, no term with exponent  $1/\Delta$  appears in Eq. (25) in agreement with the argument we presented at the beginning.

Let us now consider  $d = 3$  (an analogous argument applies to any special dimension  $d = 4 - 2/n$ ). In this case the interpretation is more difficult and we have a phenomenon analogous to the Abe-Hikami anomaly for the specific heat [13]. The origin is an additional degeneracy of the exponents for  $N \rightarrow \infty$ : the non-analytic terms with exponents  $\Delta_2/\Delta$ ,  $\Delta_3/\Delta$  and  $1 + 1/\Delta$  become degenerate with the analytic term with exponent 2. This degeneracy causes the appearance of the logarithmic term in Eq. (29) and has the consequence that the coefficients of the non-analytic terms are of  $O(1)$  instead of  $O(1/N)$  as it was the case for generic values of  $d$ . Let us write each symbol entering Eq. (17) as

$$\# = \#\infty + \frac{\#\_1}{N} + O\left(\frac{1}{N^2}\right), \quad (36)$$

and expand the  $\beta$ -function in powers of  $1/N$  to  $O(1/N)$ . Then comparing with Eqs. (19) and (29), one finds the relations

$$\begin{aligned} \alpha_{2,\infty} + \gamma_{1,\infty} + \delta_{1,\infty} + \zeta_{1,\infty} &= \frac{1}{\hat{g}_\infty^*}, \\ \delta_{1,\infty}(\Delta_{2,1} - 2\Delta_1) + \zeta_{1,\infty}(\Delta_{3,1} - 2\Delta_1) - \gamma_{1,\infty}\Delta_1 &= B(2). \end{aligned} \quad (37)$$

The leading exponent  $\Delta$  is known to  $O(1/N^2)$  for  $d = 3$  [41,42]

$$\Delta = 1 - \frac{32}{\pi^2} \frac{1}{N} - \frac{32(9\pi^2 - 80)}{3\pi^4} \frac{1}{N^2} + O\left(\frac{1}{N^3}\right). \quad (38)$$

Eqs. (37) show that at least one among the coefficients  $\gamma_1$ ,  $\delta_1$  and  $\zeta_1$  must be non-zero, and therefore that  $\alpha_{2,\infty}$ ,  $\gamma_{1,\infty}$ ,  $\delta_{1,\infty}$ , and  $\zeta_{1,\infty}$  are discontinuous at  $d = 3$ . This could be a feature of the large- $N$  limit: for finite values of  $N$  it is still possible that all coefficients be continuous in  $d$  [43].

In conclusion our explicit calculation shows that non-analytic terms are present in the  $\beta$ -function defined in Eq. (15). Notice that this result is valid for all dimensions with  $2 < d < 4$  and thus also in the  $\epsilon$ -expansion when one uses a massive renormalization scheme. In this case the singularity is of the form  $(\hat{g}^* - \hat{g})^{2/\epsilon}$ , a behaviour which has also been predicted from a large-order analysis of perturbation theory [44,45]. However, for the  $\epsilon$ -expansion, it is still possible that, as conjectured in Ref. [46], the  $\beta$ -function is analytic in a massless renormalization scheme. The question requires further investigation.

For small values of  $N$ , i.e.  $N = 0, 1, 2, 3$ , the renormalization group analysis of Ref. [47] shows that  $\Delta_2/\Delta \simeq 2$ ,  $\Delta_3/\Delta \simeq 3$ , and  $1 + 1/\Delta \simeq 3$ . The closeness of such values to integer numbers may explain the small effects of the confluent non-analytic corrections in the procedures used to estimate  $g^*$ .

## IV. $\epsilon$ -EXPANSION RESULTS

### A. Computation of $g^*$ to order $O(\epsilon^4)$

In this Section we will give the explicit expression of  $g^*$  up to four loops, i.e.  $O(\epsilon^4)$ . There are essentially two different methods to perform the calculation. One may follow the previous section: starting from the definition, one computes the renormalized two-point and four-point functions, then derives the  $\beta$ -function and finally obtains  $g^*$  from the equation  $\beta(g^*) = 0$ . Alternatively, one may compute  $g$  at three loops in terms of  $g_{\overline{\text{MS}}}$  and  $\epsilon$ . Then, to express  $g^*$  in terms of  $\epsilon$  only, one can use the four-loop expression (i.e.  $O(\epsilon^4)$ ) for the fixed point value of  $g_{\overline{\text{MS}}}$  [48]. We have followed this strategy, which allows us to obtain  $g^*$  to order  $O(\epsilon^4)$  calculating only three-loop graphs. Some intermediate results are presented in App. B. In the framework of the  $\epsilon$ -expansion, we found convenient to consider the rescaled coupling  $\bar{g}$ , defined by Eq. (6). Due to the  $O(\epsilon^{-1})$  factor multiplying  $g$  in the definition of  $\bar{g}$ , the  $O(\epsilon^4)$  of  $g^*$  corresponds to the  $O(\epsilon^3)$  of  $\bar{g}^*$ . The  $\epsilon$ -expansion of  $\bar{g}^*$  to  $O(\epsilon^3)$  is given by

$$\bar{g}^*(\epsilon) = \sum_{n=0} \bar{g}_n \epsilon^n \quad (39)$$

with

$$\bar{g}_0 = 1, \quad (40)$$

$$\bar{g}_1 = \frac{3(3N + 14)}{(N + 8)^2}, \quad (41)$$

$$\bar{g}_2 = \frac{-2864.85 - 1086.88N - 119.599N^2 - 7.07847N^3}{(N + 8)^4}, \quad (42)$$

$$\bar{g}_3 = \frac{298347 + 165854N + 38100.1N^2 + 4733.16N^3 + 240.959N^4 + 3.31144N^5}{(N + 8)^6}. \quad (43)$$

We have checked, see App. B, that these expressions reproduce the  $O(1/N)$  results of Ref. [12].

### B. Analysis of the $\epsilon$ -expansion of $\bar{g}^*$

The purpose of the calculation is, of course, the determination of  $\bar{g}^*$  in three dimensions and, possibly, in two dimensions. It is well-known that one does not obtain reliable estimates by simply setting  $\epsilon = 1$  (or  $\epsilon = 2$ ) in the corresponding series (39), since the expansion is strongly diverging.

We have analyzed the  $\epsilon$ -series using the methods proposed in Ref. [6]. The resummation technique is based on the knowledge of the large-order behaviour of the series. It is indeed known that the coefficients  $\bar{g}_n$  of the series  $\bar{g}^*(\epsilon)$  behave as

$$\bar{g}_n = c(-a)^n \Gamma(n + b_0 + 1) (1 + O(1/n)). \quad (44)$$

The constant  $a$ , which characterizes the singularity of the Borel transform of  $\bar{g}^*(\epsilon)$  does not depend on the specific observable; it is given by [49,50]

$$a = \frac{3}{N+8}. \quad (45)$$

The coefficients  $c$  and  $b_0$  depend instead on the series one considers.

Our analysis follows Ref. [6]. Given a quantity  $R$  with series  $R(\epsilon)$

$$R(\epsilon) = \sum_{k=0} R_k \epsilon^k, \quad (46)$$

we have generated new series  $R_p(\alpha, b; \epsilon)$  according to

$$R_p(\alpha, b; \epsilon) = \sum_{k=0}^p B_k(\alpha, b) \int_0^\infty dt e^{-t} t^b \frac{u(\epsilon t)^k}{(1 - u(\epsilon t))^\alpha} \quad (47)$$

where

$$u(x) = \frac{\sqrt{1+ax}-1}{\sqrt{1+ax}+1}. \quad (48)$$

The coefficients  $B_k(\alpha, b)$  are determined by the requirement that the expansion in  $\epsilon$  of  $R_p(\alpha, b; \epsilon)$  coincides with the series (46). For each  $\alpha$ ,  $b$  and  $p$  an estimate of  $R$  is simply given by  $R_p(\alpha, b; \epsilon = 1)$ .

For  $\bar{g}^*$  we have computed the series (47) for many values of  $\alpha$  and  $b$ , and for  $p = 2$  and  $p = 3$ , obtaining in this way many different estimates of  $\bar{g}^*$  in three dimensions. We have noticed that, for  $\alpha > 1$ , the estimates strongly oscillate with the number  $p$  of terms which are considered. These oscillations increase in size as  $\alpha$  increases. For this reason we have decided to keep  $\alpha$  in the interval  $-1 \leq \alpha \leq 1$ . Then for each value of  $\alpha$  and  $N$  we have considered various choices of  $b$ . In each case we have found an integer value of  $b$ ,  $b_{opt}$ , such that

$$R_3(\alpha, b_{opt}; \epsilon = 1) \approx R_2(\alpha, b_{opt}; \epsilon = 1). \quad (49)$$

In other words  $b_{opt}$  is the integer value of  $b$  that minimizes the difference between the estimates from the  $O(\epsilon^2)$  and  $O(\epsilon^3)$  series. In a somewhat arbitrary way we have then considered as our final estimate the average of  $R_p(\alpha, b; \epsilon = 1)$  with  $-1 \leq \alpha \leq 1$  and  $-2 + b_{opt} \leq b \leq 2 + b_{opt}$ .

Of course the real problem is the determination of the error bar. We have decided here to use an algorithmic procedure. The reason is that, if error and mean value are determined algorithmically, there is less chance to introduce unwillingly a systematic bias due to what we expect to be the “correct” value. Of course we do not only want to determine the error bars in a completely automatic fashion, we also want to have error bars which are reasonable. Since, as we will discuss below, we will obtain many different estimates of  $\bar{g}^*$  from the  $\epsilon$ -expansion, the basic requirement will be that all estimates should be compatible among each other. Notice that this is a requirement of internal consistency only, which does not use any external information. In this sense our procedure will be totally unbiased. Therefore discrepancies with results obtained from other methods will be meaningful. We have thus fixed our error bar as the sum of two terms: the first one is the variance of the

values of  $R_3(\alpha, b; \epsilon = 1)$  with  $-1 \leq \alpha \leq 1$  and  $\lfloor b_{opt}/2 - 1 \rfloor \leq b \leq \lceil 3b_{opt}/2 + 1 \rceil$ ; the second is the difference between the estimates from the series at order  $O(\epsilon^3)$  and  $O(\epsilon^2)$ .

To understand the reliability of our method we have reanalyzed the series of the critical exponents to order  $O(\epsilon^5)$  [48,51,52] and compared our estimates and error bars with the results of Refs. [53,54]. For  $N = 1$ , by analyzing the series of Ref. [51], the estimates  $\nu = 0.6305(25)$  and  $\gamma = 1.239(4)$  were obtained in Refs. [53,54]. Analyzing the same series by our method we find  $\nu = 0.6272(32)$  and  $\gamma = 1.2343(35)$ . These estimates agree within errors and also the error bars are very similar. Our algorithmic procedure appears to give results similar to the estimates of other authors<sup>4</sup>.

It has been noticed that the estimates of the  $\epsilon$ -expansion can be improved if one knows the (exact) value of the quantity one is considering in two dimensions [15] or even in one dimension [55]. We will now try to do more. First of all we expect  $\bar{g}^*(\epsilon)$  to be analytic in the domain  $0 < \epsilon < 4$ . This conjecture can be checked in the large- $N$  limit using the exact results of Ref. [12]. Moreover it has been implicitly assumed in the dimensional expansion around  $d = 0$  done in Refs. [56,57]. Then we will try to improve our estimates using the exactly known results for  $\epsilon = 3$  ( $d = 1$ ) and  $\epsilon = 4$  ( $d = 0$ ) and furthermore the estimates for  $\epsilon = 2$  ( $d = 2$ ).

Let us now discuss the procedure which is a simple generalization of the technique presented in Ref. [15]. Suppose that for  $\epsilon = \epsilon_1$  the exact value  $R_{\text{ex}}(\epsilon_1)$  is known. One may then define

$$\overline{R}(\epsilon) = \left[ \frac{R(\epsilon) - R_{\text{ex}}(\epsilon_1)}{(\epsilon - \epsilon_1)} \right] \quad (50)$$

and a new quantity

$$R_{\text{imp}}(\epsilon) = R_{\text{ex}}(\epsilon_1) + (\epsilon - \epsilon_1)\overline{R}(\epsilon). \quad (51)$$

New estimates of  $R$  at  $\epsilon = 1$  can then be obtained by applying the resummation procedure we described above to  $\overline{R}(\epsilon)$  and then computing  $R_{\text{imp}}(1)$ . This strategy can be generalized

<sup>4</sup> It should be noticed that the series of Ref. [51] contained an error in the five-loop coefficients as shown in Ref. [52]. An analysis of the correct series has never been published. In Ref. [52] the authors only mention that the analysis of the correct series gives results that are consistent with those of Refs. [53,54]. For the sake of completeness, we performed a new analysis of these series using our procedure. For  $N = 0, 1$  we employed a constrained analysis using the known exact results in two dimensions, as in Refs. [53,54]. We found  $\nu = 0.5882(11)$  and  $\gamma = 1.1559(10)$  for  $N = 0$ ;  $\nu = 0.631(3)$  and  $\gamma = 1.240(5)$  for  $N = 1$ . We mention that perfectly consistent results are obtained using also the homographic transformation  $\epsilon' = \lambda\epsilon/(\lambda - \epsilon)$  [6] with  $\lambda = 4, 3$  for  $N = 0, 1$  respectively. We also report results from an unconstrained analysis of the series for  $N = 2, 3$ :  $\nu = 0.664(3)$  and  $\gamma = 1.304(7)$  for  $N = 2$ ;  $\nu = 0.699(4)$  and  $\gamma = 1.372(6)$  for  $N = 3$ . For  $N = 2, 3$  this analysis seems to underestimate (slightly) the values of the exponents (see e.g. Refs. [2,60]). Indeed by employing a homographic transformation with  $\lambda = 2$  (or constraining the exact two-dimensional values:  $1/\nu = 1/\gamma = 0$ ) one obtains larger values by approximately two per cent ( $\nu \simeq 0.676$  and  $\gamma \simeq 1.324$  for  $N = 2$ , and  $\nu \simeq 0.712$  and  $\gamma \simeq 1.394$  for  $N = 3$ ), but with much larger “errors”.

to an arbitrary number of points: if exact values  $R_{\text{ex}}(\epsilon_1), \dots, R_{\text{ex}}(\epsilon_k)$  are known for a set of dimensions  $\epsilon_1, \dots, \epsilon_k$ ,  $k \geq 2$ , then one defines

$$Q(\epsilon) = \sum_{i=1}^k \left[ \frac{R_{\text{ex}}(\epsilon_i)}{(\epsilon - \epsilon_i)} \prod_{j=1, j \neq i}^k (\epsilon_j - \epsilon_i)^{-1} \right] \quad (52)$$

and

$$\bar{R}(\epsilon) = \frac{R(\epsilon)}{\prod_{i=1}^k (\epsilon - \epsilon_i)} - Q(\epsilon), \quad (53)$$

and finally

$$R_{\text{imp}}(\epsilon) = [Q(\epsilon) + \bar{R}(\epsilon)] \prod_{i=1}^k (\epsilon - \epsilon_i). \quad (54)$$

One can easily verify that the expression

$$[Q(\epsilon) + \bar{R}(0)] \prod_{i=1}^k (\epsilon - \epsilon_i) \quad (55)$$

represents the  $k$ -order polynomial interpolation among the points  $\epsilon = 0, \epsilon_1, \dots, \epsilon_k$ . Again the resummation procedure is applied to  $\bar{R}(\epsilon)$  and the final estimate is obtained computing  $R_{\text{imp}}(1)$ . The idea behind this method is very simple. If, for instance, the value of  $R$  for  $\epsilon = 2$  is known, one uses as a zeroth order approximation at  $\epsilon = 1$  the value of the linear interpolation between  $\epsilon = 0$  and  $\epsilon = 2$  and then uses the series in  $\epsilon$  to compute the deviations. If the interpolation is a good approximation one should find that the series which gives the deviations has smaller coefficients than the original one. Consequently also the errors in the resummation are reduced. In our case the value of  $\bar{g}^*$  is known for  $\epsilon = 3$  and  $\epsilon = 4$ . For  $\epsilon = 2$  we will use estimates obtained using strong-coupling methods or, for larger values of  $N$ , derived from the  $1/N$  expansion. Using this additional information we will be able to substantially reduce the error on our estimates.

As an example of the procedure let us consider the Ising model,  $N = 1$ .

In this case the series of  $\bar{g}^*(\epsilon)$  is given by

$$\bar{g}^*(\epsilon) = 1 + 0.629629\epsilon - 0.621613\epsilon^2 + 0.954535\epsilon^3 + O(\epsilon^4). \quad (56)$$

The coefficients are big, and at first sight it may seem hopeless to try to get an estimate of  $\bar{g}^*$  for  $\epsilon = 1$ . However the series alternates in sign and therefore one may obtain reasonable results after a Borel transformation. Indeed we find  $\bar{g}^* = 1.37 \pm 0.09$  which is already quite good. We can now try to improve our estimates using the value of  $\bar{g}^*$  for  $\epsilon = 2$ . Using the two-dimensional estimate  $\bar{g}^* = 1.7540(2)$  (see next Section) and Eq. (54), we get a new series for  $\bar{g}^*$ :

$$\bar{g}_{\text{imp}}^* = 1.7540 + \Delta g(1), \quad (57)$$

where

$$\Delta g(\epsilon) = -0.37698 + 0.12632\epsilon - 0.24765\epsilon^2 + 0.35345\epsilon^3 + O(\epsilon^4). \quad (58)$$

The coefficients of the new series are on average a factor of three smaller than the coefficients of the original series and the simple interpolation already gives a good estimate of  $\bar{g}_{\text{imp}}^*$ : indeed  $1.754 + \Delta g(0) = 1.377$ , not very far from the correct value. Consequently one expects a corresponding gain in the error bar. Indeed we get  $1.400 \pm 0.017$ . There is also an additional error due to the uncertainty in the two-dimensional result which in this case however turns out to be negligible. One can go further and use both the estimate for  $\epsilon = 2$  and the exact result for  $\epsilon = 3$  [12], which is  $\bar{g}^* = 9/4$ . Following the procedure we presented above, we get a new estimate of  $\bar{g}^*$  from

$$\bar{g}_{\text{imp}}^* = 1.2580 + \Delta g(1). \quad (59)$$

where

$$\Delta g(\epsilon) = 0.07937 + 0.110671\epsilon - 0.128206\epsilon^2 + 0.192895\epsilon^3 + O(\epsilon^4). \quad (60)$$

On average the coefficients of the new series are a factor of two smaller than those of the series in which only the estimate at  $\epsilon = 2$  was used. Correspondingly we obtain a slightly more precise estimate  $\bar{g}^* = 1.395 \pm 0.016$ . Again the error due to the uncertainty on the two-dimensional result is negligible.

Finally we can include also the known value of  $\bar{g}^*$  for  $\epsilon = 4$ ,  $\bar{g}^* = 3$ . Using the values at  $\epsilon = 2, 3, 4$ , we get a new expansion in the form

$$\bar{g}_{\text{imp}}^* = 1.514 + \Delta g(1). \quad (61)$$

where

$$\Delta g(\epsilon) = -0.13095 + 0.05027\epsilon - 0.08359\epsilon^2 + 0.12377\epsilon^3 + O(\epsilon^4). \quad (62)$$

Apart from the first term of the series, all the other terms have coefficients which are smaller than those of the series (60) and thus we expect an additional reduction of the error. Indeed we find  $\bar{g}^* = 1.397 \pm 0.008$ .

One can try other possibilities, constraining the series only at  $\epsilon = 3$  or  $\epsilon = 4$ , or at any possible pair. In Table I we present also the results obtained by constraining the series at  $\epsilon = 3$  and at  $\epsilon = 3, 4$ . The estimates are all consistent among each other, thus giving confidence to the final result.

We have applied this method to all values of  $N$ . In one dimension  $\bar{g}^*$  was computed in Ref. [12] finding

$$\bar{g}^* = \frac{N+8}{N+2} \left( 1 - \frac{1}{4N} \right) \quad \text{for } N \geq 1, \quad (63)$$

and

$$\bar{g}^* = \frac{N+8}{4} \quad \text{for } N \leq 1. \quad (64)$$

In zero dimensions we generalize the result of Ref. [57] to any integer  $N \geq 1$ . We get

$$\bar{g}^* = \frac{N+8}{N+2} \quad \text{for } N \geq 1. \quad (65)$$

It is not clear how to obtain the value of  $\bar{g}^*$  for  $N = 0$ . The one-dimensional results, Eqs. (63) and (64), indicate that one cannot naively set  $N = 0$  in the formula obtained for  $N \geq 1$ . The incorrectness of this analytic continuation can also be verified by our constrained analysis: if one uses the prediction  $\bar{g}^* = 4$  for  $\epsilon = 4$  one obtains results which are in total disagreement with the other estimates.

In  $d = 2$  we will use the best available estimates. For  $N \leq 4$  we consider the estimates obtained from the analysis of the strong-coupling series, more precisely,  $\bar{g}^* = 1.679(3)$ ,  $1.7540(2)$ ,  $1.810(10)$ ,  $1.724(9)$ ,  $1.655(16)$  respectively for  $N = 0, 1, 2, 3, 4$  (see next Section). For  $N \geq 8$  we use the large- $N$  expression [12]

$$\bar{g}^* = \frac{N+8}{N+2} \left( 1 - \frac{0.602033}{N} \right). \quad (66)$$

Of course the prefactor in front of the previous expression is arbitrary. However if one uses this particular form one finds a small  $1/N$  correction and good agreement up to  $N = 4$ . Indeed comparison with the (imprecise) strong-coupling result at  $N = 4$  we have reported above shows that Eq. (66) reproduces the correct result with an error smaller than 4%. In the absence of better estimates for  $N > 4$  we have thus used formula (66) with an “estimated” error of  $0.64\bar{g}^*/N^2$  (which reproduces the difference of 4% found at  $N = 4$ ).

The final results of our analysis for selected values of  $N$  are reported in Table IV.

Our results can be checked in the large- $N$  limit using the exact result,  $\bar{g}^* = 1 + 4.4540/N$  [12]. Let us consider the constrained analysis in  $d = 0, 1, 2$ , that is the one that provides the most precise estimates. In this case  $\bar{g}^*$  is determined from

$$\bar{g}^* = 1 + \frac{4.9439}{N} + \Delta g(1), \quad (67)$$

where

$$\Delta g(\epsilon) = \frac{1}{N}(-1.09695 + 0.90986\epsilon - 0.359826\epsilon^2 + 0.051144\epsilon^3). \quad (68)$$

Applying our resummation procedure to  $\Delta g(\epsilon)$  we obtain the estimate

$$\bar{g}^* = 1 + \frac{4.448(10)}{N} \quad (69)$$

in perfect agreement with the exact result reported above.

We have also repeated the analysis in two dimensions. In this case of course it is more difficult to get precise estimates: the unconstrained expansion gives results with errors of order 20 – 50% and it is therefore practically useless. Better estimates are obtained constraining the expansion in one and zero dimensions. The results for these two cases are reported in Table II, and are consistent. Of course the errors are larger than in three dimensions, but still small if one considers the series we started from (for  $N = 1$  see Eq. (56)). The resulting estimates of  $\bar{g}^*$  are in good agreement with the strong-coupling and large- $N$  estimates we used above in the analysis of the three-dimensional case, thus supporting their use.

## V. STRONG-COUPING EXPANSION

In order to understand the reliability of our previous estimates for the renormalized coupling constant, it is useful to compare them with the results of a quite different approach, such as the analysis of the high-temperature expansion for the  $N$ -vector model. In this context it is convenient to consider the coupling  $f$  defined in Eqs. (8) and (11). The finiteness of  $f^*$  is related to the hyperscaling relations of the critical exponents. A detailed analysis of the strong-coupling expansion for all values of  $N$  was presented in Ref. [12] (see also Ref. [21]) extending the numerous studies for the Ising model [17–20,22,23]. The results were in substantial agreement with the renormalization-group estimates, except for small values of  $N$ , where relatively small systematic deviations were found. The reason of these discrepancies is the presence of confluent singularities at  $\beta_c$ . Indeed, in general  $f(\beta)$  behaves as

$$f(\beta) = f^* + c_\Delta (\beta_c - \beta)^\Delta + \dots \quad (70)$$

close to the critical point. The traditional methods of analysis, like Padè and Dlog-Padè approximants, are unable to handle an asymptotic behaviour like (70) when  $\Delta$  is not an integer number, thus leading to a systematic error.

To take into account this kind of confluent corrections in full generality, one should consider integral approximants, which, however, require long series to detect non-leading effects, and in practice need to be biased to work well. Roskies proposed a rather simple method to handle the leading confluent singularity in the Ising model [32], where  $\Delta \simeq 1/2$ . He showed that the effect of the non-analytic terms can be significantly reduced by a suitable change of variables. Equivalently one can use suitably biased integral approximants [60]. In Ref. [23] the series for the renormalized coupling constant for the Ising model was analyzed biasing  $\Delta = 1/2$ . The estimated value of  $f^*$  was significantly lower than previous estimates, and now in good agreement with the renormalization-group prediction. This procedure was also successfully applied to the calculation of the critical exponent of the specific heat from the low-temperature expansion [58], providing results consistent with field theory, while a standard analysis neglecting confluent singularities led to a quite inconsistent estimate [59].

We have decided to repeat the analysis of Ref. [12] for all values of  $N$  using similar ideas. The new results show systematic differences from the old ones and now they are in much better agreement with the renormalization-group estimates.

The strong-coupling expansion of  $f(\beta)$  has the following form

$$f(\beta) = \frac{1}{\beta^{d/2}} \sum_{i=0}^{\infty} a_i \beta^i. \quad (71)$$

In three dimensions the available strong-coupling series allow us to calculate

$$A(\beta) \equiv \beta^{3/2} f(\beta) \quad (72)$$

up to 14<sup>th</sup> order (using  $\chi$  and  $m_2$  up to 15th order [60,61], and  $\chi_4$  up to 14th order [62,63]). Longer series are available for the Ising model,  $N = 1$ . On the cubic lattice, using the published series [64,65,23], one can derive  $A(\beta)$  up to 16th order. Moreover series on

other lattices are available, which allow us to calculate  $A(\beta)$  on the b.c.c., f.c.c. and diamond lattice, up to 13th, 10th, and 19th order respectively (using series published in Refs. [64,65,11,66,61]). We mention that longer series for all values of  $N$  on the cubic and b.c.c. lattice have been announced in Refs. [21,60], but they have not been published yet.

The idea of the Roskies transform (RT) [32] is to perform biased analyses which take into account the leading confluent singularity. For the Ising model, where<sup>5</sup>  $\Delta \simeq 1/2$ , one replaces the variable  $\beta$  in the original expansion with a new variable  $z$ , defined by

$$1 - z = (1 - \beta/\beta_c)^{1/2}. \quad (73)$$

Of course a quite precise estimate of  $\beta_c$  is required here. If the original series has square-root correction terms, the transformed series has analytic correction terms, which can be handled by standard Padé or Dlog-Padé approximants.

In order to analyze models with different values of  $N$  and therefore with  $\Delta \neq 1/2$ , one can generalize the Roskies transformation and consider the change of variable [33]

$$1 - z = (1 - \beta/\beta_c)^b. \quad (74)$$

In the following we will refer to this transformation as GRT. For  $b = \Delta$ , this mapping makes the first correction to scaling analytic. Thus standard approximants can handle it correctly. Of course non-analytic terms still survive due to subleading corrections, but they should be less important as far as  $\Delta$  is sufficiently smaller than the next exponents. Anyway they still represent a (hopefully small) source of systematic error for our analysis. From Eq. (74) it follows that

$$f(\beta) \longrightarrow \bar{f}(z) \equiv z^{-3/2} \bar{A}(z). \quad (75)$$

In order to estimate  $f(\beta_c) = \bar{f}(1)$  we analyze the expansion of  $\bar{A}(z)$  in powers of  $z$ . As we shall see, the use of the mapping (74) leads to a much better agreement with field-theoretic estimates.

Note that the relevant singularity of  $\bar{f}(z)$ , i.e. the one at  $z = 1$ , is no longer the closest to the origin. Indeed, the antiferromagnetic singularity at  $-\beta_c$  is mapped closer to the origin than that at  $\beta_c$ , but still at negative values of  $z$ . We expect its effect to be small when evaluating the resummed series around  $z = 1$ .

By employing standard resummation methods, we studied the behavior of  $\bar{f}(z)$  around  $z = 1$ . For the sake of comparison, we also performed a standard analysis, i.e. without using the mapping (74). We constructed various types of approximants to the series of  $\bar{A}(z)$ , such as Padé approximants (PA's), Dlog-Padé approximants (DPA's) and first-order inhomogeneous integral approximants (IA's) (for a review on the resummation techniques see for example Ref. [70]). Note that in principle IA's should be able to detect the first non-analytic correction to scaling in Eq. (70), but they probably need more terms of the series,

<sup>5</sup>We mention the recent estimate of  $\Delta$  for  $N = 1$  obtained by the fixed-dimension field-theoretic method:  $\Delta = 0.498(8)$  [67]. Notice that this is somewhat lower than the high-temperature estimate [68,69]  $\Delta = 0.54(3)$ .

and practically need to be explicitly biased as in the case of PA's and DPA's. Indeed the IA results without the GRT turn out to be substantially equivalent to those obtained from PA's and DPA's. In all cases we considered only quasi-diagonal approximants<sup>6</sup>. Then we evaluated the approximants at  $z = 1$  in order to obtain an estimate of the fixed-point value of  $f$ . Very precise values of  $\beta_c$ , which are needed for the mapping (74), are available in the literature from different calculations (see for example Refs. [71–75,60,12]). Errors due to the uncertainty on the value of  $\beta_c$  turned out to be negligible in our analysis. As estimates of  $\Delta$  we used the field-theoretic prediction for  $N \leq 24$ , and its large- $N$  expression (38) for larger values of  $N$ . Since at  $N = 0$  there is a relatively large difference between the field-theoretic prediction,  $\Delta = 0.470(25)$ , and the Monte Carlo estimate,  $\Delta = 0.515(7)^{(+10)}_{(-0)}$  [76], in our analysis we considered the very conservative value  $\Delta = 0.50(5)$ . Similarly for  $N = 1$  (see footnote 5) we consider the value  $\Delta = 0.50(5)$ .

In a PA or DPA analysis of quantities with a confluent non-analytic correction, the singularity should be mimicked by shifted poles at  $\beta \gtrsim \beta_c$ . PA's and DPA's of  $A(\beta)$  present indeed singularities typically at  $\beta \simeq 1.1 \div 1.2 \beta_c$ . On the other hand, the approximants of the series in  $z$  (cf. Eq. (74)) do not show singularities close to the new critical value  $z = 1$ , confirming the effectiveness of this change of variable. Most approximants of  $\bar{A}(z)$  do not present singularities near the real axis in the region  $\text{Re } z < 1.5$ .

Table III shows the results of our strong-coupling analysis<sup>7</sup> for selected values of  $N$ . There we report also the values of  $\beta_c$  and  $\Delta$  we used. The results of the analyses which use the GRT are quoted with two errors: the first one is the spread of the approximants for  $b = \Delta$ , the second one is due to the uncertainty on  $\Delta$ , and it is obtained by varying  $b$ . We note that for  $N = 0, 1, 2$  the estimate of  $f^*$  increases with increasing  $b$  in the GRT.

A few comments on the results of Tables III are in order.

(a) For most values of  $N$ , the difference between the results of the analysis with and without the use of the GRT, although relatively small, is larger than the apparent error of the single analysis (which is estimated by looking at the stability of the different approximants).

<sup>6</sup> Given a  $n$ th order series, we considered the following quasi-diagonal approximants:  $[l/m]$  PA's and DPA's with  $l + m \geq n - 2$  and  $l, m \geq \frac{n}{2} - 2$  ( $l, m$  are the orders of the polynomials respectively in the numerator and denominator of the PA of the series in the case of PA's, or of its logarithmic derivative in the case of DPA's);  $[m/l/k]$  IA's with  $m + l + k + 2 = n$  and  $\lfloor (n-2)/3 \rfloor - 1 \leq m, l, k \leq \lceil (n-2)/3 \rceil + 1$  ( $m, l, k$  are the orders of the polynomial  $Q_m$ ,  $P_l$  and  $R_k$  defined by the first-order linear differential equation  $Q_m(x)f'(x) + P_l(x)f(x) + R_k(x) = O(x^{k+l+m+2})$ , whose solution provides an approximant of the series at hand).

<sup>7</sup> As estimate of  $f^*$  from each class of approximants (i.e. PA's, DPA's, and IA's) we took the average of the values at  $z = 1$  (or  $\beta = \beta_c$  when performing a standard analysis) of the non-defective approximants using all the available terms of the series. The error we quote is the square root of the variance around the estimate of the results from all the non-defective approximants listed in the footnote 6. Approximants in the generic variable  $x$  are considered defective when they present spurious singularities close to the real axis for  $\text{Re } x \lesssim x_c$ . The results from PA's, DPA's, and IA's are then combined leading to the estimates shown in Table III.

This indicates that the systematic error due to the neglecting of confluent singularities is much larger than the error obtained by a stability analysis of the results.

(b) The estimates from the GRT analyses are globally in much better agreement with the field-theoretic predictions than the results obtained neglecting the confluent singularities (except for  $N = 4$ , for which the agreement of the non-biased result was already satisfactory). This can be seen in Table IV where the corresponding results for  $\bar{g}^*$  are reported. We note that the strong-coupling estimates are systematically slightly higher for  $N \geq 3$ . The uncertainty of the GRT results is approximately 1%, thus providing an accurate check of the field-theoretic calculations by a different approach.

(c) For the Ising model, universality among formulations on the cubic, b.c.c., f.c.c., and diamond lattices is well verified by the results of our GRT analysis. Assuming universality, our overall estimate is  $f^* = 23.55(15)$ . This is consistent with the result of the biased analysis (slightly different from ours) of the strong-coupling expansion on the cubic lattice of Ref. [23],  $f^* = 23.69(10)$ . We note that universality is apparently shown also by the results of the standard analysis, although they lead to a different estimate of  $f^*$ . This may be explained by noting that if the values of the non-universal coefficients  $c_\Delta$  in Eq. (70) are approximately the same for all the considered lattice formulations, they may give rise to similar systematic errors leading to an apparent universality of the results. This would not be surprising. Indeed, already the leading amplitudes of many non-universal quantities have close values in various nearest-neighbor lattice formulations (see e.g. the results reported in Ref. [22], and the estimates of some amplitudes of leading scaling corrections reported in Ref. [77]). In Ref. [22] a different analysis still neglecting confluent singularities (where the amplitudes of  $\chi$ ,  $m_2$  and  $\chi_4$  were independently calculated by a first order integral approximant analysis to give an estimate to  $f^*$ ) led to the following results:  $f^* = 24.55^{+0.95}_{-0.25}$  on the cubic lattice,  $f^* = 24.39(9)$  on the b.c.c. lattice, and  $f^* = 24.50(13)$  on the f.c.c. lattice. Universality is nicely observed, but, again, the final result is larger than the GRT estimate, and therefore also than the field-theoretic ones. Notice that these estimates of  $f^*$  are lower than our results obtained without using the GRT. This is probably due to the different procedure used to estimate  $f^*$ , and to the fact that different series were analyzed.

(d) With respect to the standard analysis, the GRT results are in much better agreement with the formula obtained by a  $1/N$  expansion [12]:

$$f^* = 16\pi \left[ 1 - \frac{1.54601}{N} + O\left(\frac{1}{N^2}\right) \right]. \quad (76)$$

This equation gives:  $f^* = 48.646$  for  $N = 48$ ,  $f^* = 47.837$  for  $N = 32$ ,  $f^* = 47.027$  for  $N = 24$ ,  $f^* = 45.408$  for  $N = 16$ , etc... This agreement provides additional support to the formal argument presented in Sec. III, according to which analytic terms should not be present in the expansion of  $f$  around  $\beta_c$ . Indeed, in the opposite case, one would not expect a substantial improvement using the mapping (74) for  $N$  sufficiently large, since  $\Delta \rightarrow 1$  for  $N \rightarrow \infty$ .

We mention that for the Ising model high-temperature techniques have also been used to obtain a dimensional expansion of the Green's functions around  $d = 0$ . The analysis of these series presented in Ref. [57] led to the quite good estimate  $f^* = 23.66(24)$ .

Let us compare our strong-coupling predictions with the results obtained from Monte Carlo simulations. Monte Carlo estimates of  $f^*$  for the Ising model on the cubic lattice can be

found in Refs. [24–31]. In Fig. 1 we compare some of the data of the most recent works with the Padé resummations of the strong-coupling series with and without the GRT. The data of Ref. [31], which are those closest to criticality, have been obtained by employing a finite-size-scaling technique, which allowed the authors to get data up to a value of  $\beta$  corresponding to  $\xi \simeq 30$ . These data show  $f(\beta)$  apparently flattened around 24.5(2). Ref. [30] presents data up to  $\xi \simeq 10$ , from which the authors obtain the value  $f^* = 25.0(5)$ . Fig. 1 shows that Monte Carlo data are in substantial agreement with our GRT analysis<sup>8</sup>. But biased approximants extrapolate to a smaller value of  $f^*$ . It is worth mentioning that the use of the GRT to bias the strong-coupling approximants is, in a sense, equivalent to the use of the function

$$f(\beta) = f^* + c_\Delta(\beta_c - \beta)^{1/2} \quad (77)$$

for the extrapolation to  $\beta_c$  of the Monte Carlo data at  $\beta < \beta_c$ . Using the function (77) to fit the Monte Carlo data of Fig. 1 one gets  $f^* = 23.7(2)$  and  $c_\Delta = 34(3)$  with  $\chi^2/\text{d.o.f} \simeq 0.6$  (here we assumed all data to be independent)<sup>9</sup>, which is perfectly consistent with the strong-coupling and field-theoretic estimate of  $f^*$ . Finally we mention the result of Ref. [29]:  $f^* = 23.3(5)$ , obtained by studying the probability distribution of the average magnetization.

We mention another application of the Roskies transform (73), that is the analysis of the low-temperature expansion of the quantity  $u$  defined in the broken phase of the Ising model by

$$u \equiv \frac{3\chi}{\xi^3 M^2}, \quad (78)$$

where  $M$  is the magnetization.  $u$  plays the important role of a zero-momentum low-temperature renormalized coupling constant in the study of the  $\phi^4$  theory directly in  $d = 3$  [78]. The most precise determinations up to now have been apparently obtained by Monte Carlo simulations [79] (where data have been fitted by using Eq. (77)):  $u^* = 14.3(1)$ , and by two different analyses of the low-temperature expansion [80,22], which lead to apparently inconsistent results:  $u^* = 14.73(14)$  [80] and  $u^* = 14.14(14)$  [22,81]. We repeated the analysis of the low-temperature expansion using the RT. The series published in Refs. [82,83] allow us to calculate the expansion of  $u$  in powers of  $e^{-4\beta}$  up to 21th order. Quasi-diagonal PA's of the Roskies-transformed series give  $u^* = 14.3(1)$  (without using the RT one obtains  $u^* = 14.7(1)$  as in Ref. [80]), thus confirming the Monte Carlo result of Ref. [79] and the low-temperature analysis of Ref. [22].

In Sec. IV we used estimates of  $f^*$  in two dimensions for our constrained analysis of the  $\epsilon$ -expansion. In the following we shortly discuss their derivations. An analysis with Padé approximants of a 17th-order series was presented by Butera and Comi [84]. For  $N \leq 2$  they

<sup>8</sup> One should also take into account that the data of Ref. [31] at different  $\beta$ 's are not statistically independent because they have been obtained by a finite-size scaling technique.

<sup>9</sup> If we do not include the data for the lowest value of  $\beta$ , corresponding to  $\xi \simeq 3$ , we obtain  $f^* = 23.9(3)$  and  $c_\Delta = 27(7)$ .

found:  $f^* = 10.53(2)$  for  $N = 0$ ,  $f^* = 14.693(4)$  for  $N = 1$ , and  $f^* = 18.3(2)$  for  $N = 2$ . We reanalyzed the same series using also DPA's and IA's. For  $N = 2$  we also considered the series in the internal energy  $E$  [12]. We found results in total agreement: 10.55(2) for  $N = 0$ , 14.693(1) for  $N = 1$  and 18.2(1) for  $N = 2$ . Moreover, using the series for the Ising model on the triangular lattice published in Refs. [64,65,85], we obtained the strong coupling expansion of  $\beta f(\beta)$  to 14th order. Its analysis gave  $f^* = 14.695(1)$ . A comparison with the square-lattice result (i.e. assuming universality) leads to the final estimate  $f^* = 14.694(2)$  for the two-dimensional Ising model.

For  $N = 3$  no estimate was reported in Ref. [84]. Our analysis gives

$$f^* = 19.7(1) \quad \text{for } N = 3. \quad (79)$$

For comparison we mention the field-theoretic estimate obtained in Ref. [86] for  $N = 3$ :  $f^* = 20.0(2)$ .

Estimates for several values of  $N > 3$  were reported in Ref. [84]. For  $N = 4$   $f^* = 20.9(1)$  was found, which is in good agreement with our reanalysis:  $f^* = 20.8(2)$ . For larger values of  $N$  — we will be interested in  $N \geq 8$  — the analysis of the strong-coupling series gives results with a somewhat large error and in this case the  $1/N$  expression [12]

$$f^* = 8\pi \left[ 1 - \frac{0.602033}{N} + O\left(\frac{1}{N^2}\right) \right], \quad (80)$$

should provide more precise estimates. Indeed it is already a good approximation for  $N = 3$  and  $N = 4$ , where it gives  $f^* = 20.09$  and  $f^* = 21.35$  respectively.

The reader should notice the small uncertainty of the estimates for  $N = 3, 4$  in spite of the fact that  $\beta_c = +\infty$ . This is due to the fact that, according to field theory,  $f(\beta)$  like any dimensionless renormalization-group invariant quantity behaves as

$$f(\beta) - f^* \sim \frac{1}{\xi(\beta)^2}, \quad (81)$$

for sufficiently large  $\beta$ . Hence the corrections to  $f^*$  decrease exponentially in  $\beta$ . This important point was overlooked in Ref. [84]. As a consequence of Eq. (81) the scaling region, where the function  $f(\beta)$  approximately reaches the asymptotic value, may begin quite early. One may obtain good estimates of the dimensionless renormalization-group invariant quantities already at  $\xi \gtrsim 10$ , which is still within the reach of the strong-coupling extrapolation, at least for not too large values of  $N$ , say  $N = 3, 4$  [87]. For instance in Fig. 2 we show  $f(\beta)$  for  $N = 3, 4$  versus the correlation length. Scaling is nicely verified by our strong-coupling calculations. Our estimates of  $f^*$  at  $N = 3, 4$  are obtained from  $f(\beta)$  at  $\xi \approx 10$ . The behaviour (81) explains the success of the strong-coupling method when applied to dimensionless renormalization-group invariant quantities (for other examples see Ref. [88]).

We finally mention the estimates of  $f^*$  for  $N = 2$  and  $N = 3$  obtained by a Monte Carlo simulation together with a finite-size scaling extrapolation [89]. For  $N = 2$ , fitting the data of Ref. [89] with  $\xi \gtrsim 10$  to a constant, we get  $f^* = 17.7(3)$ . For  $N = 3$  the result is  $f^* = 19.8(3)$ . They are in good agreement with the strong-coupling results presented above.

## VI. CONCLUSIONS

We have studied some issues concerning the fixed-point value of the zero-momentum four-point renormalized coupling  $g$  in  $O(N)$  models. The coupling  $g$  plays an important role in the field-theoretic perturbative expansion at fixed dimension, which provides an accurate description of the symmetric phase. In this approach the value of  $g^*$  is essential to compute the critical exponents, which are obtained by evaluating appropriate anomalous dimensions (calculated as functions of  $g$ ) at  $g^*$ .

The first important issue we have discussed is related to the presence of confluent singularities at the zero of the Callan-Symanzik  $\beta$ -function. In order to understand this problem we have considered the framework of the  $1/N$  expansion, which provides an analytic and non-perturbative approach. The analysis of the next-to-leading order of the  $\beta$ -function shows the presence of confluent singularities at its zero, as argued by Nickel [11]. In generic dimensions  $d \neq 4 - 2/n$  (with  $n \in \mathbb{N}$ ), the leading non-analytic corrections are  $O(1/N)$  and are related to the exponents  $\Delta_{2,3}/\Delta$  and/or  $1 + 1/\Delta$ . Since they are degenerate at  $N = \infty$ , one cannot distinguish them by an  $O(1/N)$  calculation. No term associated with an exponent  $1/\Delta$  is found. In three dimensions one meets a phenomenon analogous to the so-called Abe-Hikami anomaly [13]. One indeed finds that for  $d = 3$  the non-analytic contributions are  $O(1)$  in the  $1/N$  expansion, even if the  $\beta$ -function appears analytic to leading order. This is essentially due to a further degeneracy occurring for  $d = 3$  at  $N = \infty$  among  $\Delta_{2,3}/\Delta$ ,  $1 + 1/\Delta$  and the analytic correction with exponent two.

In the analysis of the  $g$ -expansion performed by Le Guillou and Zinn-Justin [6] with an additional hypothesis of analyticity, the presence of such singularities may cause a slow convergence to the correct fixed-point value, thus leading to an underestimation of the real uncertainty [11]. An accurate check of the  $g$ -expansion results was our major motivation for the extension of the  $\epsilon$ -expansion of  $g^*$  to  $O(\epsilon^4)$ , and for a reanalysis of the strong-coupling expansion in the lattice  $N$ -vector models.

We obtained rather accurate estimates (with an apparent precision of approximately one per cent, see Table IV) from the analysis of the 4th order  $\epsilon$ -expansion of  $g^*$  that exploits the known values for  $O(N)$  models in lower dimensions.  $g^*$  is indeed expected to be analytic in the domain  $0 < d < 4$ , as can be verified in the large- $N$  expansion to  $O(1/N)$ . We plan to extend this analysis to the low-magnetization expansion of the effective potential, which is parametrized by the zero-momentum  $n$ -point renormalized couplings.

The agreement with the  $g$ -expansion estimates is globally good. For  $N \geq 2$  there is full agreement. The results for  $N = 0, 1$  are slightly lower than the estimates given by Le Guillou and Zinn-Justin [6], thus favouring the more general analysis done by Nickel [8]. This would lead to a small change in the estimates of the critical exponents, since they depend crucially on the value of  $g^*$ . For instance, consider the case  $N = 0$ . For this model a very precise estimate<sup>10</sup> of the exponent  $\gamma$  has been recently obtained by a Monte Carlo simulation:  $\gamma = 1.1575(6)$  [90]. On the other hand, the analysis of the  $g$ -expansion, i.e. the

<sup>10</sup>It is worth mentioning that recently a very precise estimate of  $\nu$  was obtained by Belohorec and Nickel [76] by a Monte Carlo simulation of the Domb-Joyce model:  $\nu = 0.58758(7)$ .

$\beta$ -function to  $O(g^7)$  and the function  $\gamma(g)$  to  $O(g^6)$ , of Ref. [6] led to  $\gamma = 1.1615(20)$ . A more precise estimate is reported in Ref. [2],  $\gamma = 1.1607(12)$ : it is obtained using the same resummation method but one additional order in the series of  $\gamma(g)$  [8]. Reanalyzing the  $O(g^7)$  series of  $\gamma(g)$ , using the same method of Ref. [6], the authors of Ref. [90] reported the estimate

$$\gamma = 1.1616 + 0.11(\bar{g}^* - 1.421) \pm 0.0004, \quad (82)$$

where  $\bar{g}^*$  is kept arbitrary. Our result for  $\bar{g}^*$ , i.e.  $\bar{g}^* \simeq 1.39$ , thus suggests a lower value for  $\gamma$ ,  $\gamma \simeq 1.158$ , in substantial agreement with the results of the Monte Carlo simulations and with the analysis of the  $\epsilon$ -expansion (see footnote 4). Of course more precision and therefore longer series are necessary to be conclusive. We also mention that a recent analysis of the 21st order strong-coupling expansion biasing  $\Delta$  to the known approximate value has given  $\gamma = 1.1594(8)$  on the cubic lattice, and  $\gamma = 1.1582(8)$  on the b.c.c. lattice [60] (slightly larger values have been obtained by unbiased analyses).

As a by product of the  $\epsilon$ -expansion we obtained rather accurate estimates for the two-dimensional models, which are in good agreement with other estimates from lattice approaches (both strong-coupling expansion and Monte Carlo simulations) and  $1/N$ -expansion.

Finally we reanalyzed the strong-coupling expansion of three-dimensional lattice  $N$ -vector models. In order to get accurate estimates of  $g^*$ , we employed an analysis able to handle the leading confluent singularity. For this purpose we used a generalization [33] of the Roskies method [32] consisting in an appropriate change of variable. Final results have an apparent precision of approximately one per cent. We found good agreement with the field-theoretic estimates. At  $N = 0, 1$  the results seem to favor the lower values of  $g^*$  of the  $\epsilon$ -expansion.

## ACKNOWLEDGMENTS

We gratefully acknowledge useful discussions with Paolo Rossi, Lothar Schäfer and Alan Sokal. We thank Michael Fisher for useful comments on the first draft of this work.

## APPENDIX A: ASYMPTOTIC EXPANSION OF LARGE- $N$ INTEGRALS

In this appendix we give a few technical details on the computations of Sec. III.

The basic problem is the determination of the asymptotic expansion for  $\hat{g} \rightarrow \hat{g}_\infty^*$  of integrals of the form

$$I_n(f, \Theta) = \int \frac{d^d u}{(2\pi)^d} \frac{f(u^2)}{[1 + \hat{g}\Pi(u)]^n} \quad (A1)$$

where  $f(u^2)$  is a rational function of  $u^2$  and  $\Theta = (\hat{g}_\infty^* - \hat{g})/\hat{g}$ , for  $2 < d < 4$ . Using the definition of  $\Pi(u)$ , cf. Eq. (23), one can rewrite

$$I_n(f, \Theta) = \left( \frac{\hat{g}_\infty^*}{\hat{g}} \right)^n N_d \int_0^\infty u^{d-1} du \frac{f(u^2)}{[\Theta + \delta(u)]^n} \quad (A2)$$

where  $N_d$  is defined in Eq. (28) and

$$\delta(u) = \left(1 + \frac{u^2}{4}\right)^{d/2-2} {}_2F_1\left(1 - \frac{d}{2}, \frac{1}{2}; \frac{3}{2}; \frac{u^2}{4+u^2}\right). \quad (\text{A3})$$

We are interested in the large- $u$  behaviour of  $\delta(u)$ . Standard identities for the hypergeometric function give

$$\delta(u) = \left(\frac{2}{u}\right)^{4-d} c [1 + A(u)] \quad (\text{A4})$$

where  $c$  is defined in Eq. (27) and  $A(u)$  has the following asymptotic expansion

$$A(u) = \sum_{k=1}^{\infty} \frac{a_k}{u^{2k}} + \left(\frac{u}{2}\right)^{4-d} \sum_{k=1}^{\infty} \frac{b_k}{u^{2k}}; \quad (\text{A5})$$

the coefficients  $a_k$  and  $b_k$  can be easily computed for any value of  $d$ . From Eq. (A4) one immediately sees that, for  $d > 2$ ,  $\delta(u) \sim u^{d-4}$  for large values of  $u$ .

For  $d < 4$  the singularities of  $I_n(f)$  are due to the large- $u$  domain. Indeed  $\delta(u)$  goes to zero for  $u \rightarrow \infty$  and thus one cannot perform a naive expansion in powers of  $\Theta$ , since for  $k$  large enough  $I_k(f, 0)$  diverges. To compute the asymptotic expansion, let us consider

$$R(t, u; \Theta) \equiv \frac{f(t^2 u^2)}{[\Theta + t^{4-d} \delta(tu)]^n} \exp\left[c \left(\frac{2}{tu}\right)^{4-d}\right] \quad (\text{A6})$$

and its expansion for  $t \rightarrow \infty$  which has the generic form

$$R_{\text{exp}}(t, u; \Theta) = \sum_{h, k, p \geq 0} r_{h, k, p} \frac{(tu)^{-2k-p(4-d)}}{[\Theta + c(2/u)^{4-d}]^{n+h}}. \quad (\text{A7})$$

Then consider for  $2 < d < 4$

$$\int_0^\infty u^{d-1} du \left\{ \frac{f(u^2)}{(\Theta + \delta(u))^n} - R_{\text{exp}}(1, u; \Theta) \exp\left[-c \left(\frac{2}{u}\right)^{4-d}\right] \right\} \quad (\text{A8})$$

Because of the exponential factor, the second term is integrable for  $u \rightarrow 0$ . One can expand the integral (A8) in powers of  $\Theta$ . Indeed the subtraction guarantees that each term is expressed in terms of a convergent integral<sup>11</sup>. Thus the integral (A8) gives rise only to analytic contributions. The non-analytic terms can thus be computed from the subtracted term. Defining

<sup>11</sup>The reader could rightly be worried by the presence of an infinite series in (A8). Indeed all formulae should be intended in a formal sense. More precisely the procedure is the following: given  $K$ , to obtain the expansion of  $I_n(f)$  to order  $\Theta^K$ , one should consider in  $R_{\text{exp}}(t, u; \Theta)$  only those terms which, for  $\Theta = 0$ , decrease for  $u \rightarrow \infty$  less than or as  $u^{-d-K(4-d)}$ . This truncated expansion should be used in (A8).

$$x \equiv c \left( \frac{2}{u} \right)^{4-d}, \quad (\text{A9})$$

we see that we must compute integrals of the form

$$J_{n,\alpha} = \int_0^\infty dx \frac{x^\alpha e^{-x}}{(\Theta + x)^n} \quad (\text{A10})$$

with  $\alpha \geq 0$ . The computation is now trivial as

$$J_{1,\alpha} = \Theta^\alpha e^\Theta \Gamma(\alpha + 1) \Gamma(-\alpha, \Theta), \quad (\text{A11})$$

$$J_{n,\alpha} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\Theta^{n-1}} J_{1,\alpha}, \quad (\text{A12})$$

where  $\Gamma(-\alpha, \Theta)$  is the incomplete  $\Gamma$ -function [91]. For  $\Theta \rightarrow 0$  we have

$$\Gamma(-\alpha, \Theta) = \Gamma(-\alpha) - \Theta^{-\alpha} \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{n!(n-\alpha)}. \quad (\text{A13})$$

## APPENDIX B: $\epsilon$ -EXPANSION CALCULATION

In this Section we report the results of our calculation of  $g^*(\epsilon)$  to order  $O(\epsilon^4)$ ,  $\epsilon$  being defined as  $\epsilon = 4 - d$ .

The computation requires the determinations of the massive two-point and four-point functions for  $p^2 \rightarrow 0$  to three loops. The most difficult graphs are those reported in Fig. 3. The calculation is straightforward albeit long. We thus simply report the results:

$$(a) = \frac{1}{3\epsilon^3} - \frac{1}{6\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{1}{12} - \frac{\lambda}{2} + \frac{\pi^2}{24} \right) - \frac{1}{8} - \frac{\pi^2}{16} + S_1 + \frac{S_2}{4} + \frac{S_4}{4} - \frac{\zeta(3)}{6}, \quad (\text{B1})$$

$$(b) = (a), \quad (\text{B2})$$

$$(c) = \frac{1}{6\epsilon^3} + \frac{1}{\epsilon} \left( \frac{1}{24} - \frac{\lambda}{4} + \frac{\pi^2}{48} \right) + \frac{1}{24} + \frac{11\lambda}{16} + \frac{\pi^2}{24} + \frac{S_1}{2} + \frac{S_2}{8} - \frac{S_4}{4} + \frac{3S_7}{8} - \frac{5}{24}\zeta(3), \quad (\text{B3})$$

$$(d) = -\frac{5}{12\epsilon^2} - \frac{5}{48} - \frac{9\lambda}{32} - \frac{5\pi^2}{96} + \frac{5}{2}S_1 + \frac{5}{8}S_2 - \frac{3}{2}S_3 - \frac{15}{4}S_5 - \frac{15}{16}S_6 - \frac{3}{16}S_7 + \frac{3}{8}\zeta(3), \quad (\text{B4})$$

$$(e) = \frac{1}{3\epsilon^3} - \frac{1}{3\epsilon^2} + \frac{1}{\epsilon} \left( \frac{1}{12} + \frac{\pi^2}{24} \right) - \frac{\pi^2}{24} + S_3 - \frac{2}{3}\zeta(3). \quad (\text{B5})$$

Each result should be additionally multiplied by  $m^{-3\epsilon} N_d^3$  where  $N_d$  is defined in Eq. (28), and  $m$  is the mass. The constants  $\lambda$  and  $S_i$  are defined by the following integrals:

$$\lambda \equiv - \int_0^\infty dt \frac{\log t}{t^2 - t + 1} = \frac{1}{3} \psi' \left( \frac{1}{3} \right) - \frac{2\pi^2}{9} \approx 1.17195, \quad (\text{B6})$$

$$S_1 \equiv \int_0^\infty p^3 dp \log p d(p)^2 (L(p) - 2 \log p) \approx 1.0207, \quad (\text{B7})$$

$$S_2 \equiv \int_0^\infty p^3 dp d(p)^2 \int_0^1 dx [\log^2(p^2 x(1-x) + 1) - \log^2(p^2 x(1-x))] \approx -0.8619, \quad (\text{B8})$$

$$S_3 \equiv \int_0^\infty dx \left[ x K_0(x)^2 K_1(x)^2 - \frac{e^{-x/2}}{x} \left( \gamma_E + \log \frac{x}{2} \right)^2 \right] \approx 0.45077, \quad (\text{B9})$$

$$S_4 \equiv \int_0^\infty p^3 dp d(p)^2 (L(p)^2 - 4 \log^2 p) \approx 5.9622, \quad (\text{B10})$$

$$S_5 \equiv \int_0^\infty p^3 dp \log p d(p)^3 (L(p) - 2) \approx 0.18604, \quad (\text{B11})$$

$$S_6 \equiv \int_0^\infty p^3 dp d(p)^3 \int_0^1 dx \log^2(p^2 x(1-x) + 1) \approx 0.21105, \quad (\text{B12})$$

$$S_7 \equiv \int_0^\infty p^3 dp d(p)^3 (L(p) - 2)^2 \approx 0.19533, \quad (\text{B13})$$

where  $\gamma_E$  is the Euler constant,  $\gamma_E \approx 0.577216$ ,  $K_0(x)$  and  $K_1(x)$  are modified Bessel functions, and

$$d(p) \equiv \frac{1}{p^2 + 1}, \quad (\text{B14})$$

$$L(p) \equiv \xi \log \left( \frac{\xi + 1}{\xi - 1} \right), \quad (\text{B15})$$

$$\xi \equiv \sqrt{1 + \frac{4}{p^2}}. \quad (\text{B16})$$

The “Mercedes” graph ( $f$ ) requires more sophisticated techniques. To compute it, we used the method of Kotikov [92]. Let us define

$$I(M, m) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} D(p, M) D(q, M) D(r, M) D(p - q, m) D(p - r, m) D(q - r, m) \quad (\text{B17})$$

where  $D(p, M)$  is the massive free propagator with mass  $M$ . For  $m = 0$  this integral was exactly computed in Ref. [93] in all dimensions  $d$ . The expansion for  $\epsilon \rightarrow 0$  is given by

$$I(M, 0) = M^{-3\epsilon} N_d^3 \left( \frac{1}{2\epsilon} \zeta(3) - \frac{\pi^4}{80} + O(\epsilon) \right). \quad (\text{B18})$$

Using the strategy of Ref. [92] we end up with

$$I(M, M) = 3M^{-3\epsilon} \int_0^{M^2} dm^2 \frac{f(m, M) - f(0, M)}{m^2(3M^2 - m^2)} + I(M, 0) \left( 1 + \frac{3\epsilon}{2} \log \frac{3}{2} + O(\epsilon^2) \right), \quad (\text{B19})$$

where  $f(m, M)$  is the following quantity (finite for  $\epsilon \rightarrow 0$ )

$$f(m, M) = -M^2 N_d \int_0^\infty q^{d-1} dq \left[ \frac{K(m, M; q^2) J(m, M; q^2)}{q^2 + M^2} \right. \\ \left. - \frac{K(M, M; q^2) J(m, m; q^2)}{q^2 + m^2} + \frac{1}{32\pi^2} \frac{K(M, M, q^2)}{q^2 + m^2} \log \frac{m^2}{M^2} \right]. \quad (\text{B20})$$

The functions  $J$  and  $K$  can be easily computed for  $\epsilon = 0$  from the integrals

$$K(m, M; q^2) = \int \frac{d^d p}{(2\pi)^d} D(p, M)^2 D(p + q, m), \quad (\text{B21})$$

$$J(m, M; q^2) = -(mM)^{(d-4)/2} (4\pi)^{-d/2} \Gamma\left(2 - \frac{d}{2}\right) + \int \frac{d^d p}{(2\pi)^d} D(p, M) D(p + q, m). \quad (\text{B22})$$

A numerical computation gives

$$I(M, M) = N_d^3 M^{-3\epsilon} \left( \frac{1}{2\epsilon} \zeta(3) + H \right) \quad (\text{B23})$$

with  $H = -0.9825$ .

Collecting everything together, we can compute the first four coefficients in the expansion (39) of  $\bar{g}^*$ . Explicitly

$$\bar{g}_0 = 1, \quad (\text{B24})$$

$$\bar{g}_1 = \frac{3(3N + 14)}{(N + 8)^2}, \quad (\text{B25})$$

$$\begin{aligned} \bar{g}_2 = \frac{1}{(N + 8)^4} & \left[ -2N^3 + 58N^2 + 520N + 1224 - \frac{1}{3}(13N + 62)(N + 8)^2 \lambda \right. \\ & \left. - 12(5N + 22)(N + 8)\zeta(3) \right], \end{aligned} \quad (\text{B26})$$

$$\begin{aligned} \bar{g}_3 = \frac{1}{8} \frac{1}{(N + 8)^6} & \left( 4N^5 - 99N^4 + 5404N^3 + 57572N^2 + 225312N + 341312 \right) \\ & + \frac{4}{(N + 8)^5} \left( 23N^3 - 209N^2 - 2954N - 6580 \right) \zeta(3) \\ & + \frac{1}{24} \frac{1}{(N + 8)^4} \left[ (11N^3 - 4338N^2 - 45312N - 109472)\lambda + 960(2N^2 + 55N + 186)\zeta(5) \right] \\ & + \frac{1}{(N + 8)^3} \left[ -\frac{\pi^4}{15}(5N + 22) - \frac{\pi^2}{3}(19N + 62) + 16(N - 1)S_3 \right. \\ & \left. + 2(19N + 62)S_4 - \frac{1}{4}(17N^2 + 410N + 1328)S_7 - 8(5N + 22)H \right] \\ & + \frac{1}{(N + 8)^2} \left[ 24S_1 + 6S_2 + 13(N + 2)S_5 + \frac{13}{4}(N + 2)S_6 \right]. \end{aligned} \quad (\text{B27})$$

Numerical values are reported in Sec. IV A.

We can use the results for  $N \rightarrow \infty$  of Ref. [12] to check our expression. For large values of  $N$ ,  $\bar{g}^*$  is given by

$$\bar{g}^* = 1 + \frac{1}{N} \left\{ (3 - d)2^{d-1} + 8 - \int \frac{d^d u}{(2\pi)^d} \frac{\hat{g}_\infty^*}{1 + \hat{g}_\infty^* \Pi(u)} d(u)^2 \left[ 4d(u) + 9 \left( \frac{d}{2} - 1 \right) \frac{1}{u^2 + 4} \right] \right\} \quad (\text{B28})$$

where  $\hat{g}_\infty^*$  and  $\Pi(u)$  are defined in Eqs. (21) and (23) and  $d(u)$  in Eq. (B14). To obtain the series in  $\epsilon$ , we first expand the denominator in Eq. (B28) in powers of  $\hat{g}_\infty^*$  which is of order  $\epsilon$ . The computation is simple. The only integrals which require some manipulations are

$$\int \frac{d^d u}{(2\pi)^d} d(u)^2 \frac{\hat{g}_\infty^* \Pi(u)}{u^2 + 4} = \epsilon \left[ -\frac{1}{6} - \frac{\lambda}{6} + \frac{4}{9} \log 2 \right] \\ + \epsilon^2 \left[ -\frac{1}{6} - \frac{\lambda}{24} + \frac{S_5}{2} + \frac{S_6}{8} + \frac{4}{9} \log 2 - \frac{2}{9} \log^2 2 \right] + O(\epsilon^3), \quad (\text{B29})$$

$$\int \frac{d^d u}{(2\pi)^d} d(u)^2 \frac{[\hat{g}_\infty^* \Pi(u)]^2}{u^2 + 4} = \epsilon^2 \left[ -\frac{1}{6} - \frac{5\lambda}{48} + \frac{S_7}{8} + \frac{4}{9} \log 2 \right] + O(\epsilon^3). \quad (\text{B30})$$

The final result is in agreement with our expression.

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## TABLES

TABLE I. Three-dimensional estimates of  $\bar{g}^*$  from an unconstrained analysis, “unc”, and constrained analyses in various dimensions. For the analyses which use the estimates in  $d = 2$  we report two errors: the first one gives the uncertainty of the resummation of the series, the second one expresses the change in the estimate when the two-dimensional result varies within one error bar.

$N$	unc	$d = 1$	$d = 0, 1$	$d = 2$	$d = 1, 2$	$d = 0, 1, 2$
0	1.37(9)	1.39(3)		1.392(23+1)	1.390(16+1)	
1	1.37(9)	1.41(2)	1.39(3)	1.400(17+0)	1.395(16+0)	1.397(8+0)
2	1.36(7)	1.39(2)	1.41(2)	1.401(15+2)	1.411(12+3)	1.413(8+5)
3	1.35(8)	1.37(2)	1.38(2)	1.379(11+1)	1.386(7+3)	1.387(3+4)
4	1.33(5)	1.35(2)	1.36(2)	1.357(11+3)	1.364(11+6)	1.366(7+8)
8	1.29(5)	1.28(3)	1.28(2)	1.289(15+2)	1.295(4+3)	1.299(7+5)
16	1.20(2)	1.198(13)	1.196(10)	1.198(8+0)	1.198(4+1)	1.199(1+2)
24	1.154(12)	1.150(10)	1.148(8)	1.149(7+0)	1.148(5+0)	1.148(3+0)
32	1.123(10)	1.121(8)	1.119(7)	1.120(6+0)	1.118(4+0)	1.118(3+0)
48	1.087(8)	1.086(5)	1.085(5)	1.085(4+0)	1.084(3+0)	1.084(2+0)

TABLE II. Two-dimensional estimates of  $\bar{g}^*$  obtained from analyses constrained at  $d = 1$  and at  $d = 0, 1$ .

$N$	$d = 1$	$d = 0, 1$
0	1.69(7)	
1	1.79(5)	1.75(5)
2	1.75(5)	1.79(3)
3	1.68(6)	1.72(2)
4	1.61(5)	1.64(2)
8	1.45(5)	1.45(2)
16	1.28(3)	1.28(1)
24	1.21(2)	1.20(1)
32	1.16(2)	1.16(1)
48	1.11(1)	1.11(1)

TABLE III. Estimates of  $f^*$  obtained from the analysis of the available strong-coupling series with (GRT) and without (ST) the generalized Roskies transform (cf. Eq. (74)). We also report the values of  $\beta_c$  and  $\Delta$  used in our GRT analyses. The apparent uncertainty in the estimate obtained by employing the GRT is expressed as a sum of two numbers: the first number comes from the analysis at  $b = \Delta$ , the second one is due to the uncertainty on  $\Delta$ , and it is obtained by varying  $b$ . For  $N = 0, 1, 2$ , the estimate of  $f^*$  is increasing with increasing  $b$  in the GRT.

$N$	lattice	$\beta_c$	$\Delta$	GRT	ST
0	cubic	0.213492(1) [72]	0.50(5)	17.50(19+6)	19.1(4)
1	cubic	0.2216544(6) [73]	0.50(5)	23.64(14+10)	24.9(3)
1	b.c.c.	0.157373(2) [60]	0.50(5)	23.53(9+10)	24.9(2)
1	f.c.c.	0.102062(5) [22]	0.50(5)	23.53(18+10)	24.6(4)
1	diamond	0.36969(10) [61]	0.50(5)	23.47(24+12)	25.1(4)
2	cubic	0.22710(1) [74]	0.52(2)	28.45(17+5)	29.4(3)
3	cubic	0.231012(12) [75]	0.55(2)	32.24(21+5)	32.39(2)
4	cubic	0.23398(2) [60]	0.57(2)	35.10(30+10)	34.7(1)
8	cubic	0.24084(3) [60]	0.66(2)	41.50(20+10)	40.3(2)
16	cubic	0.24587(6) [60]	0.77(2)	45.81(10+10)	44.7(1)
24	cubic	0.24795(3)	0.83(2)	47.28(6+10)	46.4(1)
32	cubic	0.24907(2)	0.88(2)	47.96(4+10)	47.37(10)
48	cubic	0.25023(2)	0.93(1)	48.66(4+4)	48.32(10)

TABLE IV. Summary of the three-dimensional estimates of  $\bar{g}^*$ . In Refs. [8–10] the results were reported without errors.

$N$	$\epsilon$ -exp.	$g$ -exp.	H.T.	$1/N$ -exp.
0	1.390(17)	1.421(8) [6,2], 1.39 [8]	1.393(20)	
1	1.397(8)	1.414(3) [67], 1.40 [8]	1.406(9)	
2	1.413(13)	1.406(4) [6,2], 1.40 [8]	1.415(11)	
3	1.387(7)	1.391(4) [6,2], 1.39 [8]	1.411(12)	
4	1.366(15)	1.374 [9,10]	1.396(16)	
8	1.295(7)	1.304 [9,10]	1.321(10)	
16	1.199(3)	1.208 [9,10]	1.215(5)	1.204
24	1.148(3)	1.154 [9,10]	1.158(4)	1.151
32	1.118(3)	1.122 [9,10]	1.122(3)	1.1196
48	1.084(2)		1.084(2)	1.0839

TABLE V. Summary of the two-dimensional estimates of  $\bar{g}^*$ .

$N$	$\epsilon$ -exp.	H.T.	$1/N$ -exp.	$g$ -exp.	M.C.
0	1.69(7)	1.679(3)			
1	1.75(5)	1.7540(2)		1.85(10) [6]	1.71(12) [28]
2	1.79(3)	1.810(10)			1.76(3) [89]
3	1.72(2)	1.724(9)	1.758	1.749(16) [86]	1.73(3) [89]
4	1.64(2)	1.655(16)	1.698		
8	1.45(2)		1.479		
16	1.28(1)		1.283		
24	1.20(1)		1.200		
32	1.16(1)		1.154		
48	1.11(1)		1.106		

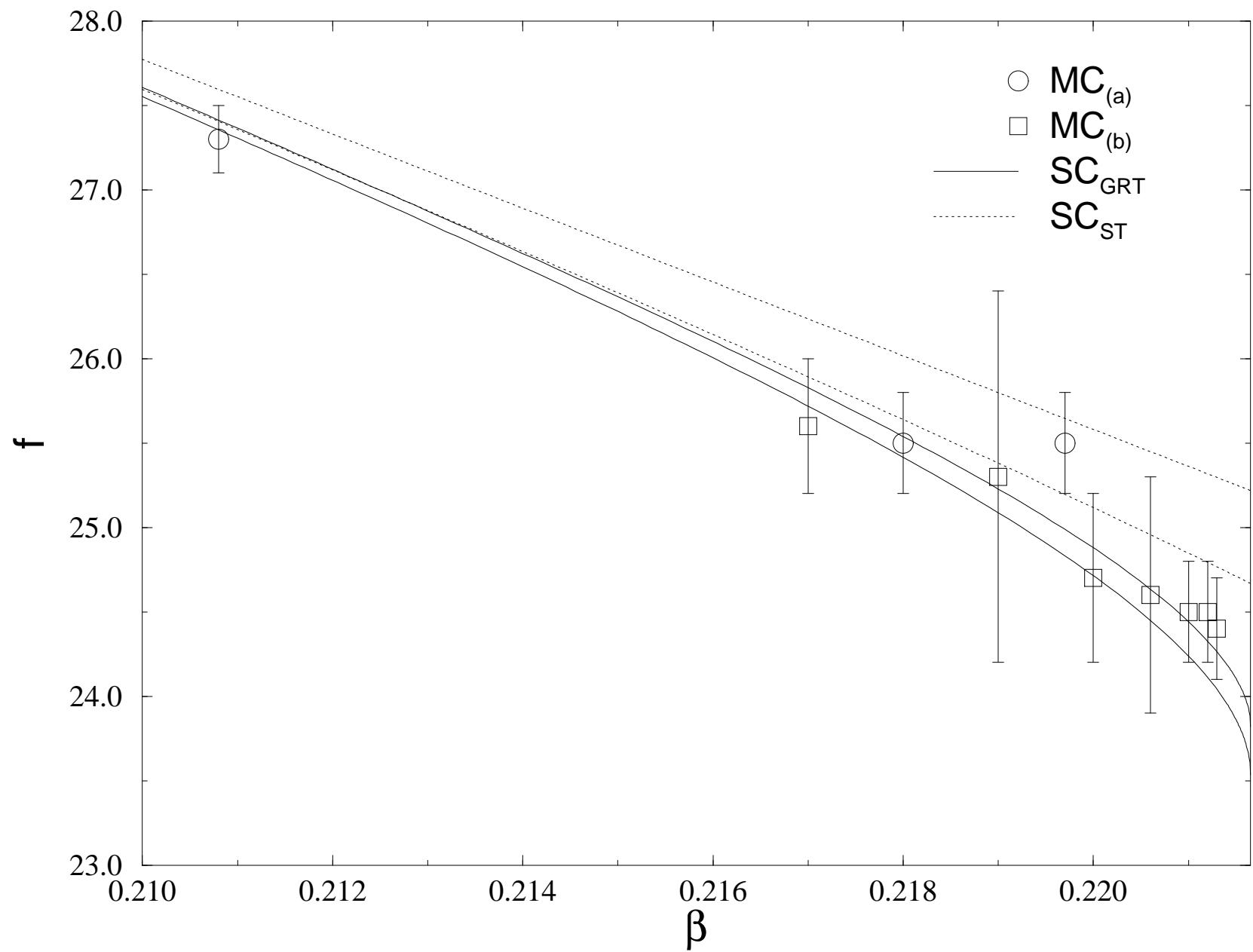
## FIGURES

FIG. 1. Plot of the function  $f(\beta)$  of the three-dimensional Ising model as obtained from the quasi-diagonal PA's calculated with and without the use of the GRT (for each case we draw two lines representing the corresponding band of uncertainty). For comparison some Monte Carlo data taken from Refs. [30] (a) and [31] (b) are also plotted.

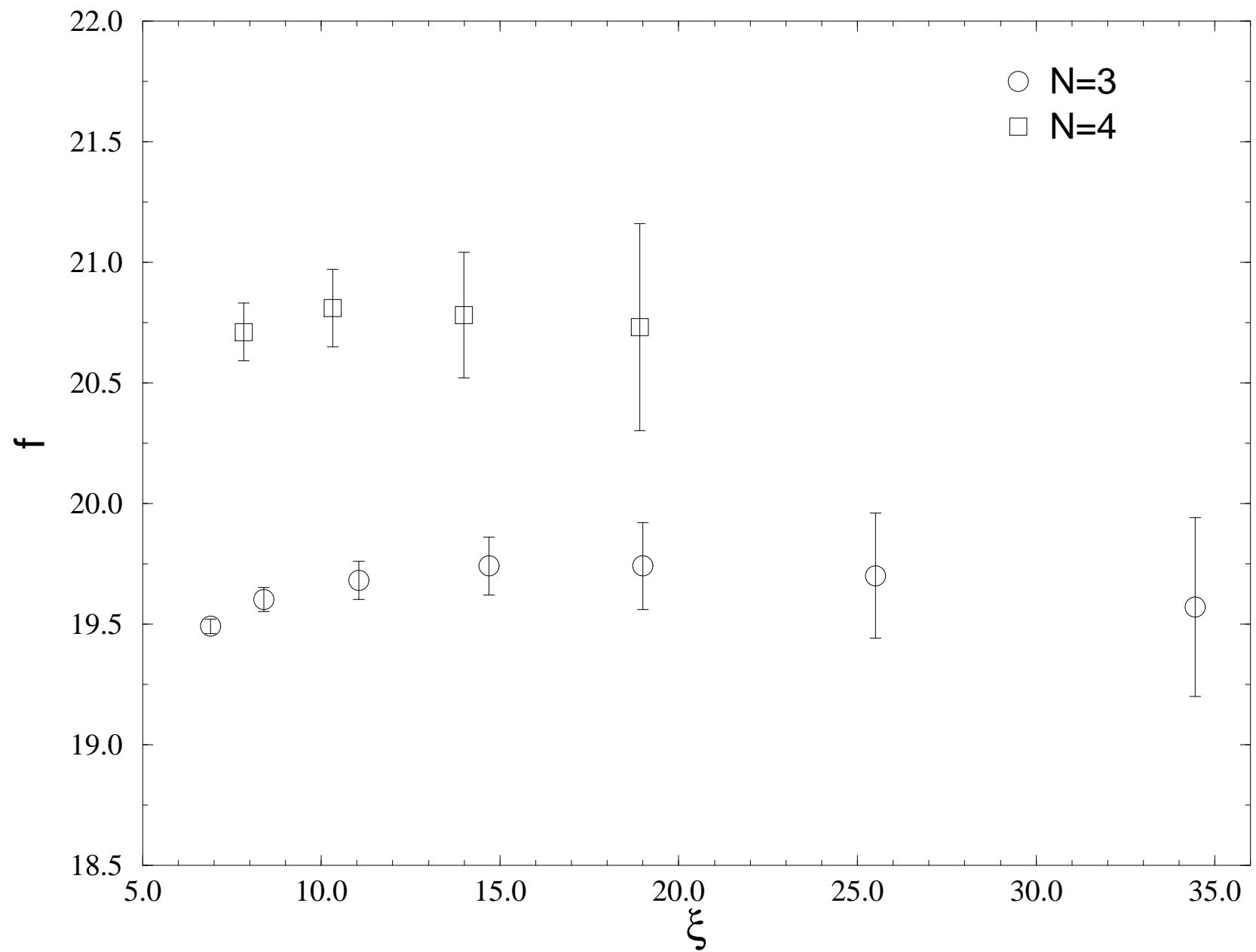
FIG. 2. Two-dimensional O( $N$ ) models with  $N = 3$  and  $N = 4$ : Plot of  $f(\beta)$  vs.  $\xi$  as obtained by the analysis of its strong-coupling series.

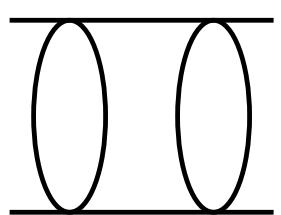
FIG. 3. Three-loop Feynman graphs contributing to the four-point correlation function.

# Figure 1

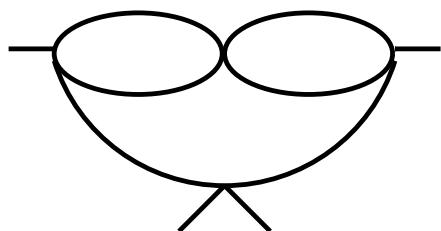


# Figure 2

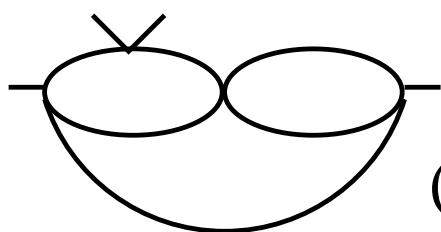




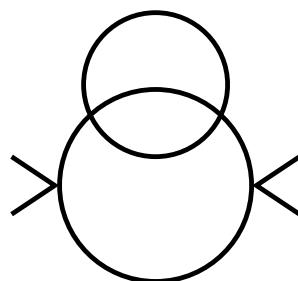
(a)



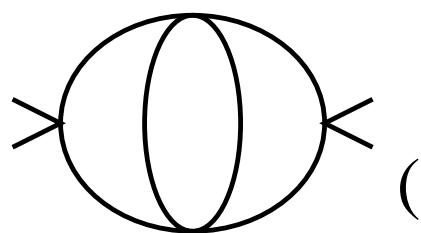
(b)



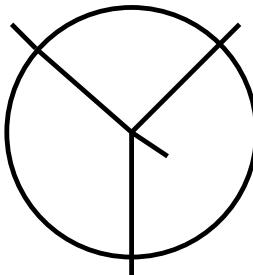
(c)



(d)



(e)



(f)